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NATIONAL BUREAU OF STANDARDS

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Section B

MATHEMATICS AND MATHEMATICAL PHYSICS

Volume 63B—July to December 1959

Papers 63B1–1 to 63B2–14

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Corrections to be noted in Volume 63 of the JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics

Page	Column	Line	Now reads in part	Should read
48	2	24	$t=0$	$Q=0$
56		21	f_ν/δ, g_ν/δ,	f_ν/δ, g_ν/δ,
59		eq. 34	$=\frac{\rho_\delta}{\rho}=\left(\text{because } \frac{\partial p}{\partial y}=0\right)$,	$=\frac{\rho_\delta}{\rho}\left(\text{because } \frac{\partial p}{\partial y}=0\right)$
		eq. 40	$+\frac{\Psi_\nu}{\delta^2}=0$	$+\frac{\Psi_\nu}{\delta_2}=0$
62		2 from bottom	Insert "with" at end of line following comma.	
77	2	3	$s+(1/n+1)2t\cos k\pi$ $(k-1$,	$s+2t\cos(k\pi/n+1)(k-1$,
	2	5	by (22)	by (16)
	2	11	$i\pm j$	$i\neq j$
82	2	18	Π Prob$\{x(t)=C, M(t)=k\|$ $x(t')=C, M(t')=kt_0<$ $t<t_1$	Π Prob $\{x(t)=C,$ $t_0<t\leq t$ $M(t)=k\|x(t')=C,$ $M(t')=k$
93	2	27	for the sweep of	for the use of
110		eq. 29	$=\xi^{-2}[G(\eta)-$	$=\eta^{-2}G(\eta)-$
115		Last	can be taken to be zero, on	can be taken to be zero, no
140		eq. 4.7	$\mathscr{I}=$	$\mathscr{I}_1=$
143		eq. 5.12	$\left\{1-\sum_{s=0}^{\infty}\frac{g_{2s+1}}{\mu^{4s+2}}\right\}_{-1}$	$\left\{1-\sum_{s=0}^{\infty}\frac{g_{2s+1}}{\mu^{4s+2}}\right\}^{-1}$

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B. MATHEMATICS AND MATHEMATICAL PHYSICS

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B. Mathematics and Mathematical Physics

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JOURNAL OF RESEARCH

The National Bureau of Standards Journal of Research reports research and development in the fields of activity shown at right. Also included from time to time are survey articles and compilations of information on topics closely related to the Bureau's scientific and technical program.

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The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to government agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. The Bureau also serves as the Federal technical research center in a number of specialized fields. The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work.

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Use of funds for printing this publication approved by the Director of the Bureau of the Budget (June 24, 1958).

Foreword

Beginning with the issue for July 1959 the Journal of Research of the National Bureau of Standards has been divided into four sections. The great diversity of work in progress at the Bureau makes such a step desirable. Section B, Mathematics and Mathematical Physics, of which this is the first issue, will be devoted to those of the areas indicated by its title in which the Bureau is active. These include not only such applied fields as numerical analysis and methods of numerical computation, compilation of mathematical tables, statistical experiment design, operations research, and the classical theories of applied mathematics, but also topics in pure mathematics which have a bearing on these fields, like functional analysis, approximation theory, combinatorial analysis, theory of matrices, inequalities, theory of games of strategy, finite projective geometries, mathematical logic, and the theories of stochastic processes and nonparametric estimation, among others. They include the theoretical foundations of fluid mechanics, elasticity, celestial mechanics, magneto-hydrodynamics, statistical mechanics and thermodynamics, electromagnetism, optics, and nuclear physics, as well as the underlying mathematical treatment of differential and integral equations and special functions. Finally, they include the logical design and programing of digital computers.

In these and related fields, section B of the Journal will be devoted primarily to reporting on research at the Bureau. In addition, there will occasionally be invited papers by authorities outside the Bureau dealing with related subjects. We also plan to present occasional review articles, tables of useful data, and compilations of information on subjects closely related to the Bureau's program. Thus, section B is planned to include some of the material which heretofore the Bureau has been publishing in its nonperiodical series of publications. A complete listing, with selected abstracts, of NBS publications and of articles by the Bureau staff in professional journals will also be a regular feature.

C. H. Page, *Editor*,
Section B. Mathematics and Mathematical Physics.

Journal of Research of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 63B, No. 1, July–September 1959

Relations Between Summation Methods and Integral Transformations [1]

Werner Greub [2]

(May 12, 1959)

Relations between the Lototsky method of summation and those of Borel, Euler, and Knopp are obtained by associating an integral transformation with the series transformation.

1. Introduction

In a recent paper, R. P. Agnew [1] [3] establishes relations between the Lototsky method of summation for divergent series and the classical summations of Borel, Euler, and Knopp.

It is the purpose of this paper to show that these relations can be obtained in a very natural way if the series transformation is associated with an integral transformation. The following scheme describes the connections between the series and the integral transformation:

original series	⟶	transformed series
↑ Laurent expansion		↑ Taylor expansion
analytic function	⟶	integral transform of the function

In section 2 it is shown that under certain conditions a regular series transformation can be assigned to every integral transformation, and conversely. As an example, the integral transformation corresponding to the Lototsky series transformation (L-transformation) is constructed. This integral transformation is used in section 3 for the discussion of the relation between the Lototsky and the Borel integral methods. The outcome of this discussion is that L-summability contains Borel integral-summability (BI-summability) but does not contain a modified Borel summability (BI*-summability as denoted in [1]). It is shown that the L-method can be extended to a summation method which is equivalent to the BI*-method.

In section 4 it is proved that a power series is L-summable in all interior points of the Borel polygon of the corresponding analytic function but not L-summable in all the exterior points. This proof is based on the relationship between the L-method and the BI-method.

In section 5 the Lototsky method is compared with the method of Euler-Knopp.

In the last section the inverse of the L-integral transform is constructed and used to obtain the inverse matrix of the Lototsky series transformation.

[1] This work was carried out in part under a National Bureau of Standards contract with The American University.
[2] Present address: University of Zürich, Zürich, Switzerland.
[3] Figures in brackets indicate the literature references at the end of this paper.

2. General Relations Between Series and Integral Transforms

Let $a_{n\nu}$ $(n,\nu=1,2,\ldots)$ be an infinite matrix whose elements are complex and subject to the condition

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|a_{n\nu}|}=0 \qquad (n=1,2,\ldots). \tag{1}$$

We consider a not necessarily convergent infinite series

$$\sum_{\nu=1}^{\infty} u_\nu \tag{2}$$

with the property

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|u_\nu|}<\infty. \tag{3}$$

This condition assures that the radius of convergence of the power series

$$\sum_{\nu=1}^{\infty} \frac{u_\nu}{z_\nu}$$

is positive.

Under conditions (1) and (3) all the series

$$U_n=\sum_{\nu=1}^{\infty} a_{n\nu}u_\nu$$

are convergent, as can readily be seen with the help of the root test. Therefore, we can say that under those conditions the matrix $(a_{n\nu})$ defines the series transformation

$$u_n \to U_n=\sum_{\nu=1}^{\infty} a_{n\nu}u_\nu. \tag{4}$$

We will show now that one can associate with this series transformation a certain integral transformation between the analytic functions

$$f(z)=\sum_{\nu=1}^{\infty} \frac{u_\nu}{z^\nu} \tag{5}$$

and

$$\phi(\alpha)=\sum_{n=1}^{\infty} U_n\alpha^n.$$

To obtain the kernel of this transformation we define the functions $P_n(z)$ by

$$P_n(z)=\sum_{\nu=0}^{\infty} a_{n\,\nu+1}z^{\nu} \qquad (n=1, 2, \ldots).$$

Because of (1) these series are convergent for every z and thus $P_n(z)$ are entire functions of z.

It follows from (3) that the function $f(z)$ is regular outside of a certain circle, $|z|=\rho$. Now we integrate the product, $P_n(z)f(z)$ along a circle $|z|=R \quad (R>\rho)$, and obtain

$$\oint P_n(z)f(z)dz=\oint \sum_{\nu,\mu} a_{n\,\nu+1}u_\nu z^{\nu\mu}dz$$

$$=2\pi i\sum_{\mu=1}^{\infty} a_{n\,\mu}u_\mu=2\pi i U_n. \qquad (6)$$

Under the hypothesis that the function

$$K(z,\alpha)=\sum_{n=1}^{\infty} P_n(z)\alpha^n$$

is regular in the domain $|z|<\infty$, $|\alpha|<1$, we derive from (6)

$$\oint K(z,\alpha)f(z)dz=2\pi i\sum_{n=1}^{\infty} U_n\alpha^n.$$

Hence, defining the function $\phi(\alpha)$ for $|\alpha|<1$ by

$$\phi(\alpha)=\sum_{n=1}^{\infty} u_n\alpha^n,$$

we obtain the integral transformation [4]

$$\phi(\alpha)=\frac{1}{2\pi i}\oint K(z,\alpha)f(z)dz, \qquad (7)$$

which maps every function $f(z)$ which is regular outside of a certain circle into a function $\phi(\alpha)$ which is regular within the unit circle.

Conversely, let (7) be an integral transformation whose kernel $K(z,\alpha)$ is regular in the domain $|z|<\infty$, $|\alpha|<1$ and fulfills the condition [5] $K(z,0)=0$. Substituting for $f(z)$ a function [6] regular for $|z|>1$,

$$f(z)=\sum_{\nu=1}^{\infty}\frac{u_\nu}{z^\nu},$$

we obtain for $\phi(\alpha)$ a function which can be expanded into a power series within the unit circle,

$$\phi(\alpha)=\sum_{n=1}^{\infty} U_n\alpha^n.$$

From (7) we obtain between the coefficients u_ν and U_n the relations

$$U_n=\frac{1}{2\pi i}\oint\frac{\phi(\alpha)}{\alpha^{n+1}}d\alpha=\left(\frac{1}{2\pi i}\right)^2\oint\oint\oint\frac{K(z,\alpha)}{\alpha^{n+1}z^\nu}f(z)dzd\alpha$$

$$=\left(\frac{1}{2\pi i}\right)^2\oint\oint\oint\sum_{\nu}\frac{K(z,\alpha)}{\alpha^{n+1}}\frac{u_\nu}{z^\nu}dzd\alpha. \qquad (8)$$

Hence the integral transformation (7) defines a series transformation whose matrix is given by

$$a_{n\nu}=\left(\frac{1}{2\pi i}\right)^2\oint\oint\oint\frac{K(z,\alpha)}{\alpha^{n+1}z^\nu}dzd\alpha.$$

It can be shown that under certain hypotheses about the kernel the series transformation (8) is regular in the following sense: If the series $\sum_\nu u_\nu$ is convergent, the series $\sum_n U_n$ is Abelian summable (A-summable). This means that the limit

$$\lim_{\alpha\to 1-0}\left(\sum_{n=1}^{\infty} U_n\alpha^n\right) \qquad (\alpha \text{ real})$$

exists and the relation

$$\lim_{\alpha\to 1-0}\left(\sum_{n=1}^{\infty} U_n\alpha^n\right)=\sum_{\nu=1}^{\infty} u_\nu$$

holds. The hypotheses about the kernel are:

(i) $K(z,\alpha)$ is a regular function of z and α in the domain $|z|<\infty$, $|\alpha|<1$ and therefore can be expanded in the form

$$K(z,\alpha)=\sum_{n=0}^{\infty} a_{n+1}(\alpha)z^n. \qquad (9)$$

(ii) For every fixed n we have

$$\lim_{\alpha\to 1-0} a_n(\alpha)=1 \qquad (\alpha \text{ real}).$$

(iii) There exists a constant M not depending on α such that

$$\sum_{n=1}^{\infty}|a_{n+1}(\alpha)-a_n(\alpha)|\leqq M.$$

To prove the above statement we consider the transformation

$$\phi(\alpha)=\frac{1}{2\pi i}\oint K(z,\alpha)f(z)d(z)$$

and substitute for $K(z,\alpha)$ and $f(z)$ the expansions (9) and (5), respectively. So we obtain the series-to-function transformation

$$\phi(\alpha)=\sum_{\nu=1}^{\infty} u_\nu a_\nu(\alpha). \qquad (10)$$

[4] The path of integration in the z-plane will always be a circle $|z|=R \quad (R>1)$.

[5] This is not an essential restriction because it can be achieved by replacing $K(z,\alpha)$ by $K(z,\alpha)-K(z,0)$. Then the function $\phi(\alpha)$ is only changed by a constant.

[6] It is assumed that there is no constant term in the expansion; in other words, that $f(\infty)=0$.

Using a well-known regularity theorem we conclude from conditions (ii) and (iii) that this transformation is regular; this means, the limit $\lim_{\alpha\to 1-0}\phi(\alpha)$ exists provided that the series $\sum_v u_v$ is convergent and the relation

$$\lim_{\alpha\to 1-0}\phi(\alpha)=\sum_{v=1}^{\infty}u_v. \tag{11}$$

holds.

The statement which has just been proved can be formulated as the following:

THEOREM 1. *Let $K(z,\alpha)$ be a function with the properties (i), (ii), and (iii), and let the matrix a_{nv} be defined by*

$$a_{nv}=\left(\frac{1}{2\pi i}\right)^2\oint\oint\frac{K(z,\alpha)}{\alpha^{n+1}z^v}\,dzd\alpha.$$

Then the series transformation induced by this matrix carries every convergent series $\sum_v u_v$ into an A-summable series $\sum_n U_n$ with the same sum.

It is worth noting that the series $\sum_n U_n$ need not be convergent itself. However, if the matrix a_{nv} fulfills additional inequalities of the form

$$\sum_{v=1}^{\infty}|a_{nv}|\leqq\frac{M}{n}\qquad(n=1,2,\ldots) \tag{12}$$

where M is a fixed number, it follows that

$$|U_n|\leqq\frac{M}{n}.$$

In this case we can apply the Tauberian theorem and conclude that the series $\sum_n U_n$ is convergent and

$$\sum_n U_n=\lim_{\alpha\to 1-0}\sum_n(U_n\alpha^n).$$

Thus, under the additional hypothesis (12), we get

$$\sum_n U_n=\sum_v u_v$$

for every convergent series $\sum_v u_v$, which means that the transformation (4) is regular.

As an example, let us consider the series transformation

$$U_1=u_1,\quad U_n=\frac{1}{n\,!}\sum_{v=1}^{n}p_{n-1,v-1}u_v \tag{13}$$

where the p_{nv} are the coefficients of the polynomial

$$p_n(z)=z(z+1)\ldots(z+n-1)\qquad(n=1,2,\ldots). \tag{14}$$

This transformation was introduced by Lototsky [3] and will be called the L-transformation. If the series $\sum_n U_n$ is convergent respectively A-summable the original series $\sum_v u_v$ will be called L-summable respectively AL-summable, and we write

$$\sum_n U_n=L\left(\sum_v u_v\right)$$

and

$$\lim_{\alpha\to 1-0}\left(\sum_n U_n\alpha^n\right)=AL\left(\sum_v u_v\right).$$

For the integral kernel, $K(z,\alpha)=\sum_{n=1}^{\infty}P_n(z)\alpha^n$, of the L-transformation, the functions $P_n(z)$ have the following form:

$$P_1(z)=z$$

$$P_n(z)=\frac{1}{n!}\,p_{n-1}(z)=\frac{(-1)^n}{z-1}\binom{1-z}{n}\qquad(n=2,3,\ldots). \tag{15}$$

This leads to

$$K(z,\alpha)=\frac{1}{z-1}\sum_{n=1}^{\infty}(-1)^n\binom{1-z}{n}\alpha^n=\frac{(1-\alpha)^{1-z}-1}{z-1}.$$

Hence, defining the analytic function $\phi(\alpha)$ ($|\alpha|<1$) by

$$\phi(\alpha)=\frac{1}{2\pi i}\oint\frac{(1-\alpha)^{1-z}-1}{z-1}\,f(z)dz, \tag{16}$$

the AL-summability of the series $\sum_v u_v$ means that $\phi(\alpha)$ approaches a limit if α tends to 1 from the left along the real axis and we have the relation

$$AL\left(\sum_v u_v\right)=\lim_{\alpha\to 1-0}\phi(\alpha).$$

In order to show that a convergent series is also AL-summable to the same value we check that the kernel $K(z,\alpha)$ satisfies the hypothesis of theorem 1. Condition (i) is immediately clear and (ii) follows from the fact that $a_n(\alpha)$ can be written as

$$a_n(\alpha)=\frac{1}{2\pi i}\oint\frac{K(z,\alpha)}{z^n}\,dz=1-e^{-\gamma}\sum_{v=1}^{n}\frac{\gamma^v}{v!} \tag{17}$$

where

$$\gamma=-\log(1-\alpha).$$

To check (iii) we observe that

$$a_{n+1}(\alpha)-a_n(\alpha)=e^{-\gamma}\,\frac{\gamma^{n+1}}{(n+1)!},$$

and therefore

$$\sum_{n=1}^{\infty} |a_{n+1}(\alpha)-a_n(\alpha)| = e^{-\gamma} \sum_{n=1}^{\infty} \frac{\gamma^{n+1}}{(n+1)!}$$

$$< e^{-\gamma}e^{\gamma} = 1 \qquad (0 \leqq \alpha < 1).$$

Now it follows from theorem 1 that the transformation (13) maps a convergent series into an A-summable series with the same sum.

In this special case it even can be shown that the series $\sum_n U_n$ is convergent itself. From (13) we get

$$\sum_{\nu=1}^{\infty} a_{n\nu} = \frac{1}{n!} \sum_{\nu=1}^{n} p_{n-1}p_{\nu-1} = \frac{1}{n!}\, p_{n-1}(1) = \frac{1}{n},$$

and thus condition (12) is fulfilled which assures the convergence of the series $\sum_n U_n$. This means that the L-method is regular.

3. Connections Between the Lototsky and the Borel Summabilities

As mentioned before, there is a close relationship between the Lototsky and the Borel-integral methods of summation. Let $\sum_\nu u_\nu$ be a (not necessarily convergent) infinite series such that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{|u_n|}{n!}} = 0. \tag{18}$$

Then the sum

$$F(t) = \sum_{\nu=0}^{\infty} \frac{u_{\nu+1}}{\nu!}\, t^{\nu} \tag{19}$$

defines an entire function of t. The series $\sum_\nu u_\nu$ is called *summable with respect to the Borel integral method* (BI-summable) if the improper integral

$$\int_0^{\infty} e^{-t}F(t)\,dt$$

exists. In this case we write

$$BI\left(\sum_\nu u_\nu\right) = \int_0^{\infty} e^{-t}F(t)\,dt.$$

We are going to show that the function

$$B(t) = \int_0^t e^{-\tau}F(\tau)\,d\tau \tag{20}$$

can be represented as a certain integral transform of the function

$$f(z) = \sum_{\nu=1}^{\infty} \frac{u_\nu}{z^\nu}. \tag{21}$$

Assume (as we did in sec. 2) that the radius of convergence of the series (21) is positive; in other words, that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|u_n|} = \frac{1}{r} < \infty. \tag{22}$$

Since $\sqrt[n]{n!} \to \infty$ it follows from (22) that condition (18) is automatically fulfilled.

The coefficients u_n are represented by the integrals

$$u_{n+1} = \frac{1}{2\pi i} \oint f(z)z^n\,dz \tag{23}$$

where the integral is taken around a certain circle $|z| = R \quad (R > r)$. From (23) it follows that

$$\sum_{n=0}^{\infty} \frac{u_{n+1}}{n!}\, t^n = \frac{1}{2\pi i} \oint f(z) \sum_{n=0}^{\infty} \frac{z^n t^n}{n!}\, dz = \frac{1}{2\pi i} \oint f(z)e^{zt}\,dz,$$

and therefore, using (19) and (20),

$$B(t) = \frac{1}{2\pi i} \int_0^t \oint e^{\tau(z-1)} f(z)\,dz\,d\tau. \tag{24}$$

Since the integrand is regular we can interchange the two integrations and integrating with respect to τ we obtain

$$B(t) = \frac{1}{2\pi i} \oint \frac{e^{t(z-1)}-1}{z-1}\, f(z)\,dz. \tag{25}$$

Thus the BI-sum of the series $\sum_\nu u_\nu$ can be represented as the limit

$$BI\left(\sum_\nu u_\nu\right) = \lim_{t\to\infty} \frac{1}{2\pi i} \oint \frac{e^{t(z-1)}-1}{z-1}\, f(z)\,dz. \tag{26}$$

Comparing transformations (16) and (25), we obtain the following relation between functions $\phi(\alpha)$ and $B(t)$:

$$\phi(\alpha) = B(-\log(1-\alpha)) \qquad (|\alpha| < 1), \tag{27}$$

which can also be written as

$$B(t) = \phi(1-e^{-t}). \tag{28}$$

From this relation it follows immediately that the existence of one of the limits

$$\lim_{\alpha\to 1-0} \phi(\alpha) \quad \text{or} \quad \lim_{t\to\infty} B(t)$$

implies the existence of the other, and these limits coincide. This means in terms of series: *Under the hypothesis (22) AL- and the BI-summability are equivalent.*

Here the question arises whether this equivalence still holds without condition (22). We are going to show that the implication BI→AL is true in general but not conversely.

We first prove

LEMMA 1: *Let u_v be an infinite series such that the function*

$$F(t)=\sum_{v=0}^{\infty}\frac{u_{v+1}}{v!}\,t^v$$

is regular in a certain neighborhood of $t=0$ and define the functions $B(t)$ and $\phi(\alpha)$ by

$$B(t)=\int_0^t F(\tau)e^{-\tau}d\tau$$

and

$$\phi(\alpha)=B(-\log\,(1-\alpha)).$$

Then the coefficients U_n of the expansion

$$\phi(\alpha)=\sum_n U_n\alpha^n$$

are the L-transforms of u_v.

PROOF: The coefficients U_n are defined by

$$B(-\log\,(1-\alpha))=\sum_n U_n\alpha^n.$$

Differentiating this relation we get

$$B'(-\log\,(1-\alpha))\,\frac{1}{1-\alpha}=\sum_n nU_n\alpha^{n-1}, \tag{29}$$

and hence, observing that

$$B'(t)=F(t)e^{-t}$$

and therefore

$$B'(-\log(1-\alpha))=(1-\alpha)F(-\log\,(1-\alpha)),$$

we conclude that

$$\sum_n nU_n\alpha^{n-1}=\sum_{v=0}^{\infty}\frac{(-1)^v}{v!}\,[\log\,(1-\alpha)]^v u_{v+1}.$$

Using the residue theorem we conclude that

$$U_n=\frac{1}{n}\sum_{v=0}^{\infty}\frac{(-1)^v}{v!}\left(\oint\frac{[\log\,(1-\alpha)]^v}{\alpha^{n+1}}\,d\alpha\right)u_{v+1},$$

which can be written as

$$U_n=\frac{1}{n!}\sum_{v=0}^{\infty}a_{n-1\,v}u_{v+1}=\frac{1}{n!}\sum_{v=1}^{\infty}a_{n-1\,v-1}u_v, \tag{30}$$

where the coefficients a_{nv} are defined by

$$a_{nv}=(-1)^v\frac{n!}{v!}\oint\frac{[\log\,(1-\alpha)]^v}{\alpha^{n+1}}\,d\alpha. \tag{31}$$

It has to be shown that the numbers a_{nv} coincide with the numbers p_{nv} defined by (14). First of all we have $a_{nv}=0$ for $v>n$ since the functions

$$\frac{[\log\,(1-\alpha)]^v}{\alpha^{n+1}}$$

are regular at $\alpha=0$ provided that $v\geqq n+1$. We now define the polynomials $a_n(z)$ by

$$a_n(z)=\sum_{v=1}^{n}a_{nv}z^v$$

and show that

$$a_n(-k)=0\qquad(k=0,1,\ldots,n-1).$$

In fact we have for every integer

$$\frac{1}{n!}\,a_n(-k)=\sum_{v=1}^{n}\frac{k^v}{v!}\oint\frac{[\log\,(1-\alpha)]^v}{\alpha^{n+1}}\,d\alpha$$

$$=\sum_{v=1}^{\infty}\frac{k^v}{v!}\oint\frac{[\log\,(1-\alpha)]^v}{\alpha^{n+1}}\,d\alpha=\oint\frac{(1-\alpha)^k-1}{\alpha^{n+1}}\,d\alpha.$$

If now $k\leqq n-1$ there is no residue at $\alpha=0$ and we get

$$a_n(-k)=0\qquad(k=0,1,\ldots,n-1). \tag{32}$$

From here it follows, since the polynomial $a_n(z)$ has the degree n,

$$a_n(z)=\lambda z(z+1)\ \ldots\ (z+n-1) \tag{33}$$

where λ is a constant. Substituting $z=1$ in (33) we get

$$a_n(1)=\lambda n!$$

and on the other hand we have

$$a_n(1)=\sum_{v=1}^{n}a_{nv}=\sum_{v=1}^{\infty}a_{nv}=n!\sum_{v=1}^{\infty}\frac{(-1)^v}{v!}\oint\frac{[\log\,(1-\alpha)]^v}{\alpha^{n+1}}\,d\alpha$$

$$=n!\oint\left(\frac{1}{1-\alpha}-1\right)\frac{d\alpha}{\alpha^{n+1}}=n!.$$

From the last two relations it follows that $\lambda=1$ and we obtain

$$a_n(z)=z(z+1)\ \ldots\ (z+n-1)=p_n(z)$$

and comparing the coefficients

$$a_{nv}=p_{nv},$$

which proves lemma 1.

THEOREM 2.[7] *A BI-summable series is also AL-summable and the two sums coincide:*

$$BI\left(\sum_v u_v\right)=AL\left(\sum_v u_v\right).$$

PROOF: By hypothesis the series

$$F(t)=\sum_{v=0}^{\infty}\frac{u_{v+1}}{v!}\,t^v$$

defines an entire function. Hence the function

$$B(t)=\int_0^t F(\tau)e^{-\tau}\,dt$$

is entire too, and therefore the function

$$\phi(\alpha)=B(-\log(1-\alpha)) \tag{34}$$

is regular for $|\alpha|<1$. According to lemma 1 we have the expansion

$$\phi(\alpha)=\sum_{n=1}^{\infty}U_n\alpha^n \tag{35}$$

where the U_n are the L-transforms of the u_v. Hence the series

$$\sum_{n=1}^{\infty}U_n\alpha^n$$

is convergent for $|\alpha|<1$. Furthermore, we obtain from (34) and (35)

$$\sum_{n=1}^{\infty}U_n\alpha^n=B(-\log(1-\alpha)),$$

and hence, if α tends to 1 on the real axis,

$$\lim_{\alpha\to 1-0}\left(\sum_{n=1}^{\infty}U_n\alpha^n\right)=\lim_{t\to+\infty}B(t).$$

This means that the series $\sum_v u_v$ is AL-summable and that the relation

$$AL\left(\sum_v u_v\right)=BI\left(\sum_v u_v\right)$$

holds.

Conversely, one cannot conclude that an AL-summable (even an L-summable) series is also BI-summable, as the following example shows:

Let the function $\varphi(t)$ be defined by

$$\varphi(t)=\frac{1}{2-e^{-t}}$$

and the sequence u_n by

$$u_n=\varphi^{(n-1)}(0)\qquad(n=1,2,\ldots). \tag{36}$$

Then we have

$$\sum_{v=0}^{\infty}\frac{u_{v+1}}{v!}\,t^v=\frac{1}{2-e^{-t}},$$

and therefore this series is convergent only for $|t|<\log 2$. Hence the BI-method cannot be applied.

To obtain the L-transform of sequence (36) we apply lemma 1. The function $B(t)$ in our case is given by

$$B(t)=\int_0^t\frac{e^{-\tau}d\tau}{2-e^{-\tau}}=\log(2-e^{-t}),$$

and from here we get

$$\phi(\alpha)=B(-\log(1-\alpha))=\log(1+\alpha),$$

and thus

$$U_n=(-1)^{n-1}\frac{1}{n}.$$

The series $\sum_n U_n$ is convergent, and hence series (36) is L-summable to the value log 2.

As the above example shows, the AL-summability of a series $\sum_v u_v$ does not imply that the series

$$\sum_v\frac{u_{v+1}}{v!}\,t^v$$

has an infinite radius of convergence. It can be shown however that this radius can not be less than log 2. To this purpose we prove

LEMMA 2: *Let $\sum_v u_v$ be an infinite series and $\sum_n U_n$ its L-transform. Assume that the series*

$$\phi(\alpha)=\sum_v U_n\alpha^n$$

is convergent in a certain circle $|\alpha|<\rho$, $(\rho>0)$. Then the series

$$F(t)=\sum_v\frac{u_{v+1}}{v!}\,t^v$$

is convergent in a certain circle $|t|<\rho'$, $(\rho'>0)$ and the relation

$$\int_0^t\left(\sum_v\frac{u_{v+1}}{v!}\,\tau^v\right)e^{-\tau}\,d\tau=\phi(1-e^{-t}) \tag{37}$$

holds.

PROOF: By hypothesis the function

$$B(t)=\phi(1-e^{-t}) \tag{38}$$

is regular in a certain circle $|t|<\rho'$ and therefore the same is true for the function

$$F(t)=B'(t)e^t, \tag{39}$$

[7] Different proofs of theorems 2 to 4 are given in a mimeographed report by R. P. Agnew (U.S.A.F. Contract 18 (600-685)).

where B' indicates the derivative of B.

From (38) and (39) it follows that

$$F(t)=\phi'(1-e^t). \tag{40}$$

Expanding the function $F(t)$ into a power series,

$$F(t)=\sum_v c_v t^v,$$

we obtain from (40)

$$\phi'(1-e^{-t})=\sum_v c_v t^v,$$

and if we introduce

$$\alpha=1-e^{-t}$$

as a new variable,

$$\phi'(\alpha)=\sum_v (-1)^v c_v [\log(1-\alpha)]^v. \tag{41}$$

On the other hand we have

$$\phi'(\alpha)=\sum_n n U_n \alpha^{n-1}. \tag{42}$$

From (41) and (42) it follows that

$$\sum_n n U_n \alpha^{n-1}=\sum_v (-1)^v c_v [\log(1-\alpha)]^v,$$

and from here we obtain

$$U_n=\frac{1}{2\pi i}\frac{1}{n}\sum_v (-1)^v c_v \oint \frac{[\log(1-\alpha)]^v}{\alpha^n}\,d\alpha$$

$$=\frac{1}{n!}\sum_v \rho_{n-1\,v}\, v! c_v.$$

This equation states that the U_n are the L-transforms of the sequence $v!c_v$. On the other hand, they are by definition the L-transforms of u_v, and hence we conclude

$$c_v=\frac{1}{v!}u_{v+1}.$$

Therefore the expansion of $F(t)$ becomes

$$F(t)=\sum_{v=0}^{\infty}\frac{u_{v+1}}{v!}t^v \tag{43}$$

and this series must be convergent within the circle $|t|<\rho'$.

Relation (37) now follows from (43), (39), and (38).

From lemma 2 we can derive the following theorem, which is proved by Agnew [2] in a different way.

THEOREM. *For the terms of an AL-summable series* $\sum u_v$ *the inequality*

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\frac{|u_n|}{n!}}\leqq\frac{1}{\log 2} \tag{44}$$

holds.

PROOF: Since the series $\sum u_v$ is AL-summable the function

$$\phi(\alpha)=\sum_n U_n \alpha^n$$

is regular for $|\alpha|<1$. Hence the function $B(t)$ defined by

$$B(t)=\phi(1-e^{-t})$$

must be regular in the domain D, which is determined by the inequality

$$D\colon |1-e^{-t}|<1; \tag{45}$$

and the same holds for the function

$$F(t)=B'(t)e^{-t}.$$

Writing $t=x+iy$, we get

$$|1-e^{-t}|^2=1+2e^{-x}(\tfrac{1}{2}e^{-x}-\cos y)$$

and hence inequality (45) is equivalent to

$$\tfrac{1}{2}e^{-x}<\cos y.$$

This means that the point $t=x+iy$ lies in the "interior" of the curve

$$x=-\log(2\cos y). \tag{46}$$

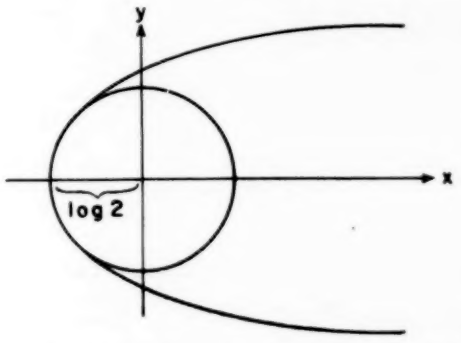

FIGURE 1. *Relations between summation methods and integral transformations.*

Thus the function $B(t)$ can be expanded into a power series within the largest circle $|t|=\rho$ with the center $t=0$ that is contained in the domain D. This is the circle

$$|t|=\log 2.$$

because this circle touches curve (46) in the point $t=-\log 2$, it is sufficient to show that the radius of curvature of the curve (46) is not less than log 2. The curvature of this curve is given by

$$\frac{1}{\rho}=\frac{1}{\cos y},$$

and hence we have

$$\rho > 1 > \log 2.$$

Thus the expansion of $F(t)$ into a power series must be convergent for

$$|t| < \log 2.$$

On the other hand, using lemma 2 this expansion is given by

$$F(t) = \sum_v \frac{u_{v+1}}{v!} t^v.$$

From here it follows that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{|u_n|}{n!}} \leqq \frac{1}{\log 2},$$

which proves theorem 3.

It follows from series (36) that the bound $1/\log 2$ can not be improved.

The BI-method.* The application of the BI-method provides that the series

$$\sum_v \frac{u_{v+1}}{v!} t^v$$

converges in the whole t-plane. Frequently the case occurs that the radius of convergence of this series is positive and finite such that this series defines a regular function $F(t)$ within a certain circle $|t| < \rho$. If this function can be analytically continued along the positive t-axis one can form the improper integral

$$\int_0^\infty F(t) e^{-t} dt.$$

In case this integral is convergent, the series is said to beBI*-summable [2, sec. 11] and its value is called the BI*-sum of $\sum_v u_v$,

$$BI^*\left(\sum_v u_v\right) = \int_0^\infty F(t) e^{-t} dt.$$

Obviously a BI-summable series is also BI*-summable to the same value.

It can be shown that AL-summability implies BI*-summability, but we do not prove this here since BI*-summability will turn out to be equivalent to a generalization of AL-summability and the above statement will follow immediately from there.

To demonstrate the BI*-method by an example, let u_n be defined by

$$u_{n+1} = (-1)^n \frac{n!}{a^n} \qquad (n = 0, 1, 2, \ldots) \qquad (47)$$

where a $(a \neq 0)$ is an arbitrary complex number. In this case we obtain

$$\sum_v \frac{u_{v+1}}{v!} t^v = \frac{1}{1 + t/a} = \frac{a}{a+t},$$

and thus,

$$BI^*\left(\sum_v u_v\right) = \int_0^\infty \frac{a e^{-t}}{a+t} dt.$$

This integral converges for every number a which is not real and negative and therefore series (47) is BI*-summable for all those values of a.

The next example shows that a BI*-summable series is not necessarily AL-summable (and hence not L-summable). Let the function $\phi(t)$ be defined by

$$\phi(t) = \frac{1}{(a - e^{-t})^2}, \qquad (48)$$

where a is a real number such that $a > 1$, and let the sequence u_n be defined by

$$u_n = \phi^{(n-1)}(0) \qquad (n = 1, 2, \ldots). \qquad (49)$$

Then the series

$$\sum_v \frac{u_v}{(v+1)!} t^v$$

is convergent in the circle

$$|t| < log\ a$$

and represents the function $\phi(t)$. This function is defined along the whole positive t-axis and the integral

$$\int_0^\infty \frac{e^{-t} dt}{(a - e^{-t})^2} = \frac{1}{a-1} - \frac{1}{a}$$

is convergent. Thus the series (49) is BI*-summable.

In order to obtain the L-transform of (49) according to lemma 2 we have to form the function

$$B(t) = \int_0^t \phi(\tau) e^{-\tau} d\tau = \frac{1}{a-1} - \frac{1}{a - e^{-t}}$$

and then substitute

$$t = -\log\ (1 - \alpha).$$

This gives

$$\phi(\alpha) = \frac{1}{a-1} - \frac{1}{a - 1 + \alpha}.$$

The function $\phi(\alpha)$ so obtained has a pole at $\alpha = 1 - a$ and hence the series $\sum_n U_n \alpha^n$ is only convergent in the circle

$$|\alpha| < a - 1,$$

which is smaller than the unit circle if $a<2$. Therefore, the AL-method can not be applied to series (49).

The AL-summability.* As the last example shows the BI*-method is more powerful than the AL-method. However the AL-method can be extended to a method AL* in a similar way as BI has been extended to BI*.

Let $\sum_v u_v$ be an infinite series such that the power series

$$\phi(\alpha)=\sum_n U_n \alpha^n$$

is convergent within a circle $|\alpha|<\rho$ and assume that the function $\phi(\alpha)$ can be analytically continued along the interval $0\leqq u<1$ $(\alpha=u+iv)$. If the limit

$$\lim_{\alpha\to 1-0} \phi(\alpha) \qquad (\alpha \text{ real})$$

exists we shall call the series $\sum_v u_v$ AL*-summable and write

$$\mathrm{AL}^*\Big(\sum_v u_v\Big)=\lim_{\alpha\to 1-0}\phi(\alpha).$$

AL*-summability obviously implies AL-summability.

THEOREM 4. *The AL*-summability is equivalent to the BI*-summability.*

PROOF: Assume first the series $\sum_n u_v$ is BI*-summable. Then the function

$$F(t)=\sum_v \frac{u_{v+1}}{v!}\, t^v$$

is regular within a circle $|t|<\rho$. Let now the functions $B(t)$ and $\phi(\alpha)$ be given by

$$B(t)=\int_0^t F(\tau)e^{-\tau}d\tau$$

and

$$\phi(\alpha)=B(-\log(1-\alpha)). \tag{50}$$

Then $\phi(\alpha)$ is regular within a circle $|\alpha|<\rho'$ and therefore it can be expanded into a power series

$$\phi(\alpha)=\sum_n U_n \alpha^n.$$

According to lemma 1 the coefficients U_n are the L-transforms of the u_n. By hypothesis, the function $F(t)$ can be analytically continued along the positive t-axis and therefore the same is true for the function $B(t)$. Hence eq (50) defines an analytic continuation of the function $\phi(\alpha)$ along the real unit interval and we have

$$\lim_{\alpha\to 1-0}\phi(\alpha)=\lim_{t\to\infty} B(t).$$

This means that the series $\sum_v u_v$ is *AL**-summable and the relation

$$\mathrm{AL}^*\Big(\sum_v u_v\Big)=\mathrm{BI}^*\Big(\sum_v u_v\Big)$$

holds.

On the other hand, if the series $\sum_v u_v$ is AL*-summable, the function

$$\phi(\alpha)=\sum_n U_n\alpha^n$$

is regular within a circle $|\alpha|<\rho$. Hence the series

$$F(t)=\sum_v \frac{u_{v+1}}{v!}\, t^v$$

according to lemma 2 is convergent within a circle $|t|<\rho'$, and the relation

$$F(t)=\phi'(1-e^{-t})e^t \tag{51}$$

holds. Since the function $\phi(\alpha)$ by hypothesis can be analytically continued along the real unit interval the same holds for $F(t)$ along the positive t-axis and relation (51) holds for every positive t. From there we obtain

$$\int_0^\infty F(t)e^{-t}dt=\lim_{\alpha\to 1-0}\phi(\alpha)$$

which shows that the series $\sum_v u_v$ is BI*-summable.

Altogether we have the following relations between the Lototsky and the Borel methods, where all the simple arrows are not invertable:

$$\begin{array}{ccccc} L & \to & \mathrm{AL} & \to & \mathrm{AL}^* \\ & & \uparrow & & \downarrow\uparrow \\ & & \mathrm{BI} & & \mathrm{BI}^* \end{array}$$

Series (36) shows that L does not imply BI, but it is not known to the author whether the implication BI→L holds.

3. Application to Analytic Continuation

Let $g(z)$ be an analytic function regular in a domain D which contains the origin and let

$$g(z)=\sum_{v=1}^{\infty} u_v z^{v-1} \tag{52}$$

be the expansion of $g(z)$ around the point $z=0$. It is a known result that the series (52) is BI-summable to the value $g(z)$ at every interior point of the Borel polygon π of $g(z)$ and not BI-summable at every exterior point. Making use of the relationship between the BI- and the L-summability the same can be shown for the L-summability of series (52).

In order to do so we first observe that

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|u_n z_0^{n-1}|}=|z_0|\overline{\lim_{n\to\infty}}\sqrt[n]{|u_n|}<\infty,$$

and hence the BI-summability is equivalent to AL-summability (sec. 3). From this remark and the implication L→AL it follows that series (52) can not be L-summable at an exterior point of π.

Let now z_0 be an interior point of π. Then the series $\sum_r u_r z_0^{r-1}$ is BI-summable and hence AL-summable. Thus the relation

$$\lim_{\alpha\to 1-0}\left(\sum_n U_n(z_0)\alpha^n\right)=g(z_0) \tag{53}$$

holds where $U_n(z_0)$ denotes the L-transform of the sequence $u_r z_0^{-1}$.

It remains to show that the series $\sum_n U_n(z_0)$ is convergent; then the relation

$$\sum_n U_n(z_0)=g(z_0)$$

will follow from (53) and the Abelian limit theorem.

We are going to prove the absolute convergence of the above series. For that we start out from the relations

$$U_n(z_0)=\frac{1}{n!}\sum_{r=1}^{n-1} p_{n-1\,r}\, u_{r+1}\, z_0^r. \tag{54}$$

Since z_0 is an interior point of the polygon π the closed circle

$$\left|z-\frac{z_0}{2}\right|\leqq\frac{|z_0|}{2}$$

is contained in D, and therefore a number $\rho>1$ can be chosen so close to 1 that the circle

$$c: \qquad \left|z-\frac{z_0}{2}\right|=\rho\,\frac{|z_0|}{2}$$

still is contained in D. [2, Chap. VIII.]

Hence the coefficients u_r can be represented by

$$u_r=\frac{1}{2\pi i}\oint\frac{g(z)}{z^r}\,dz. \tag{55}$$

From (54) and (55) we obtain

$$U_n=\frac{1}{n!}\oint_c p_{n-1\,r}\,\frac{g(z)}{z^{r+1}}\,z_0^r\,dz=\frac{1}{n!}\oint_c\frac{g(z)}{z}\,p_{n-1}\left(\frac{z_0}{z}\right)dz$$

and from there, if M denotes the maximum of $\frac{g(z)}{z}$ on the circle c,

$$|U_n|\leqq\frac{M}{n!}\left|p_{n-1}\left(\frac{z_0}{z}\right)\right|.$$

If we write for abbreviation

$$\omega=\frac{z_0}{z},$$

this relation becomes

$$|U_n|\leqq\frac{M}{n!}\,|\omega||\omega+1|\ldots|\omega+n-2|,$$

$$=M\left|\frac{\omega(\omega+1)\ldots(\omega+n)}{n!n^\omega}\right|\cdot\left|\frac{n^2}{(\omega+n-1)(\omega+n)}\right|\cdot|n^{\omega-2}|. \tag{56}$$

If n tends to ∞ the first factor approaches $\Gamma(\omega)$ and therefore stays bounded. The same is true for the second term and therefore it is sufficient to show that the series

$$\sum_n|n^{\omega-2}| \tag{57}$$

is uniformly convergent on c. The circle c is represented by the equation

$$z=\frac{z_0}{2}\,(1+\rho e^{i\theta})\qquad(0\leqq\theta\leqq 2\pi).$$

From here it follows that

$$\omega=\frac{z_0}{z}=\frac{2}{1+\rho e^{i\theta}}$$

and for the real part

$$R(\omega)=\frac{2(1+\rho\cos\theta)}{1+2\rho\cos\theta+\rho^2}\leqq\frac{2}{1+\rho}.$$

Therefore we have

$$R(\omega-2)\leqq\frac{-2\rho}{1+\rho}=-1-\delta,\qquad(\delta>0)$$

and hence the inequalities

$$|n^{\omega-2}|\leqq n^{-(1+\delta)}$$

hold, which imply the uniform convergence of series (57) on c. Now the absolute convergence of the series $\sum_n U_n$ follows from (56).

5. Connections With the Euler-Knopp Method

The Euler-Knopp transform of a sequence u_r (EK-transform) is defined to be the sequence

$$U_n=\sum_r\binom{n-1}{r-1}\kappa^r(1-\kappa)^{n-r}\,u_r \tag{58}$$

where κ is a complex parameter.

This transformation can be derived from a linear transformation of the w-plane which has the fixed

point $\omega=1$ and maps $\omega=0$ into $\omega=\infty$. The most general transformation of this kind is given by

$$z=\frac{1-a}{\omega}+a \tag{59}$$

where a is a complex number. Now let

$$f(z)=\sum_v \frac{u_v}{z^v} \tag{60}$$

be an analytic function which is regular for $|z|>1$. This implies that the function

$$f^*(\omega)=f\left(\frac{1-a}{\omega}+a\right) \tag{61}$$

is regular in the domain that is mapped onto $|z|>1$ under the transformation (59), that is the circle

$$|1-a+a\omega|>|\omega|.$$

Since this domain contains the point $\omega=0$ the function $f^*(\omega)$ can be expanded into a power series

$$f^*(\omega)=\sum_n U_n \omega^n.$$

The coefficients U_n are given by

$$U_n=\frac{1}{2\pi i}\oint \frac{f^*(\omega)}{\omega^{n+1}}\,d\omega.$$

If we substitute here for the function $f^*(\omega)$ the expression (61) and for $f(z)$ the expansion (60) we get

$$U_n=\frac{1}{2\pi i}\sum_v \oint \left(\frac{\omega}{1-a+a\omega}\right)^v \frac{d\omega}{\omega^{n+1}}\,u_v.$$

This integral can be evaluated by the residue theorem. First we observe that

$$(1-a+a\omega)^{-v}=(1-a)^{-v}\left(1+\frac{a}{1-a}\right)^{-v}$$
$$=\kappa^v\{1+(\kappa-1)\omega\}^{-v}$$

where

$$\kappa=\frac{1}{1-a}. \tag{62}$$

From there we obtain

$$U_n=\frac{1}{2\pi i}\left(\oint \sum_{v,\,\mu} \kappa^v \binom{-v}{\mu}(\kappa-1)^\mu \omega^{v+\mu-n-1}\,d\omega\right)u_v$$
$$=\sum_v \binom{-v}{n-v}\kappa^v(\kappa-1)^{n-v}u_v$$

and making use of the identity

$$\binom{-v}{n-v}(-1)^{n-v}=\binom{n-1}{v-1},$$

we finally get

$$U_n=\sum_v \binom{n-1}{v-1}\kappa^v(1-\kappa)^{n-v}u_v.$$

This equation shows that the coefficients U_n are the EK-transforms of the coefficients u_v.

From the relationshup between the EK-transformation and the linear mapping (59) it can easily be derived that the EK-transformation cannot be regular unless κ is real and $0<\kappa\leqq 1$. Assume that transformation (58) is regular and let $\sum_v u_v$ be a convergent series such that the function

$$f(z)=\sum_v \frac{u_v}{z_v} \tag{63}$$

has a singularity at a given point z_1 ($|z_1|=1$, $z_1\neq 1$). Then the series $\sum_n U_n$ by hypothesis is also convergent and hence the function

$$f^*(\omega)=f\left(\frac{1-a}{\omega}+a\right) \tag{64}$$

is regular for $|\omega|<1$. Therefore the function $f(z)$ must be regular in the image domain of the circle $|\omega|<1$ under the mapping (59). This image is given by

$$D_a:\ |z-a|>|1-a|.$$

Therefore the point z_1 cannot be contained in D_a and since z_1 was arbitrary on the unit circle this has to hold for the whole circle $|z|=1$. Hence the unit circle has to be contained in the circle

$$|z-a|\leqq|1-a|$$

and this is possible only if a is real and $a\leqq 0$, which means that κ is real and $0<\kappa\leqq 1$.

It is known that an EK-summable series is also AL-summable provided that κ is real and positive [2, sec. 8.] We shall give here a proof of this fact using the integral representation of the L-transformation.

Let $\sum_v u_v$ be an EK-summable series and define the function $f(z)$ by

$$f(z)=\sum_v \frac{u_v}{z_v}.$$

Then the function

$$g(\omega)=f^*\left(\frac{1}{\omega}\right)=f((1-a)\omega+a)$$

has the expansion

$$g(\omega)=\sum_n U_n\omega^n.$$

The function $f(z)$ can be expressed by $g(\omega)$ as

$$f(z)=g\left(\frac{z-a}{1-a}\right).$$

From this equation it follows that the L-integral transform of $f(z)$ can be written as

$$\phi(\alpha)=\frac{2}{2\pi i}\oint\frac{(1-\alpha)^{1-z}-1}{z-1}f(z)dz$$

$$=\frac{1}{2\pi i}\oint\frac{(1-\alpha)^{1-z}-1}{z-1}g\left(\frac{z-a}{1-a}\right)dz.$$

Introducing the new variable

$$\zeta=\frac{z-a}{1-a}$$

we obtain

$$\phi(\alpha)=\frac{1}{2\pi i}\oint\frac{(1-\alpha)^{(1-\zeta)(1-a)}-1}{\zeta-1}g(\zeta)d\zeta$$

and hence $\phi(\alpha)$ can be considered as the transform of $g(\omega)$ with respect to the kernel

$$\overline{K}(\zeta,\alpha)=\frac{(1-\alpha)^{(1-\zeta)(1-\alpha)}-1}{\zeta-1}.$$

It is easily verified that this kernel fulfills the conditions of theorem 1 provided that a is real and $0\leqq a<1$. Therefore it follows that under this condition

$$\lim_{\alpha\to 1-0}\phi(\alpha)=\sum_n U_n=\mathrm{EK}\left(\sum_\nu u_\nu\right)$$

provided that the series $\sum_n U_n$ is convergent. But that means that the series $\sum_\nu u_\nu$ is AL-summable and the relation

$$\mathrm{AL}\left(\sum_\nu u_\nu\right)=\mathrm{EK}\left(\sum_\nu u_\nu\right)$$

holds.

In case κ is not both real and positive it is shown by Agnew ([2], sec. 8) that EK-summability does not imply AL-summability.

6. Inversion Formula of the Lototsky Integral Transformation

The question arises whether the integral transformation

$$\phi(\alpha)=\frac{1}{2\pi i}\oint\frac{(1-\alpha)^{1-z}-1}{z-1}f(z)dz \tag{65}$$

can be inverted. One cannot expect that to every given regular function $\phi(\alpha)$ ($|\alpha|<1$) there exists a function $f(z)$ regular outside of a sufficiently large circle such that relation (65) holds. Provided such a function exists, it can be represented as an integral transform of $\phi(\alpha)$ at least in a half-plane.

THEOREM 5. *Let $f(z)$ be a function regular for $|z|>r$ such that $f(\infty)=0$, and let $\phi(\alpha)$ be its transform with respect to (65). Then in the half-plane $x>r$ the function $f(z)$ can be represented as*

$$f(z)=(z-1)\int_0^1\phi(\alpha)(1-\alpha)^{z-2}d\alpha.$$

PROOF: Since $f(\infty)=0$, relation (65) can be written as

$$\phi(\alpha)=\frac{1}{2\pi i}\oint\frac{(1-\alpha)^{1-z}}{z-1}f(z)dz. \tag{66}$$

Let $z_0=x_0+iy_0$ be a fixed point in the half-plane $x>r$. We choose for the path of integration in (65) a circle $|z|=R$, where

$$r<R<x_0.$$

If (65) is multiplied by $(1-\alpha)^{z_0-2}$ and integrated with respect to α between 0 and t ($0<t<1$), we get

$$\int_0^t\phi(\alpha)(1-\alpha)^{z_0-2}d\alpha=\frac{1}{2\pi i}\int_0^t\oint\frac{(1-\alpha)^{z_0-z-1}}{z-1}f(z)dzd\alpha.$$

Since the integrand is regular, the two integrations may be interchanged and we obtain integrating with respect to α

$$\int_0^t\phi(\alpha)(1-\alpha)^{z_0-2}\,d\alpha=-\frac{1}{2\pi i}\oint\frac{(1-t)^{z_0-z}-1}{(z-1)(z_0-z)}f(z)dz.$$

If t tends to 1, the expression

$$(1-t)^{z_0-z}$$

tends to zero uniformly on $|z|=R$ because of the inequality

$$\mathrm{Re}\ (z_0-z)\geqq x_0-R>0.$$

Hence we obtain

$$\int_0^1\phi(\alpha)(1-\alpha)^{z_0-2}\,d\alpha=\frac{1}{2\pi i}\oint\frac{f(z)dz}{(z-1)(z_0-z)}. \tag{67}$$

The right-hand side of this equation can be evaluated by the Cauchy integral formula since the function $f(z)$ is regular for $|z|>r$. Observing that $f(\infty)=0$, we get

$$\frac{1}{2\pi i}\oint \frac{f(z)dz}{(z-1)(z_0-z)}=\frac{f(z_0)}{z_0-1}. \tag{68}$$

From (67) and (68) it follows that

$$\frac{f(z_0)}{z_0-1}=\int_0^1 \phi(\alpha)(1-\alpha)^{z_0-2}d\alpha$$

and we obtain the formula

$$f(z)=(z-1)\int_0^1 \phi(\alpha)(1-\alpha)^{z-2}d\alpha \qquad (x>r) \tag{69}$$

which is the inversion of (65).

Formula (69) can be used to obtain the inverse matrix of the L-transformation (13). For the particular functions

$$\phi(\alpha)=\alpha^k \qquad (k=0, 1 \ldots) \tag{70}$$

we get from (69)

$$f(z)=(z-1)\int_0^1 \alpha^k(1-\alpha)^{z-2}d\alpha$$
$$=\frac{k!}{z(z+1)\ldots(z+k-1)}=\frac{k!}{p_k(z)}.$$

This function is in fact regular outside of the circle $|z|=k-1$ and thus for the special functions (70) the integral equation (65) does have a solution that is regular outside of a certian circle.

To obtain the inverse of the L-transformation

$$U_n=\frac{1}{n!}\sum_v p_{n-1\,v-1}u_v, \tag{71}$$

we denote by q_{nj} the inverse of the matrix p_{nj}. Then it follows from (71) that

$$u_n=\sum_j j!q_{n-1\,j-1}U_j. \tag{72}$$

The functions

$$\phi(\alpha)=\sum_n U_n\alpha^n$$

and

$$f(z)=\sum_r \frac{u_r}{z^r}$$

correspond to each other by the transformations (65) and (68).

In particular, the function (70) possesses the coefficients

$$U_n=\delta_{nk},$$

and hence we obtain from (72)

$$u_n=\sum_j j!q_{n-1\,j-1}\delta_{jk}=k!q_{n-1\,k-1}.$$

Thus the function $f(z)$ has the expansion

$$f(z)=k!\sum_n \frac{q_{n-1\,k-1}}{z^n}. \tag{73}$$

On the other hand this function is given by

$$f(z)=\frac{k!}{p_k(z)} \tag{74}$$

Comparing the right-hand sides of (73) and (74) we conclude that

$$\sum_n \frac{q_{n-1\,k-1}}{z^n}=\frac{1}{p_k(z)}$$

and from there

$$q_{n-1\,k-1}=\frac{1}{2\pi i}\oint \frac{z^{n-1}}{p_k(z)}\,dz$$

or if n−1 is replaced by n and $k-1$ by k,

$$q_{nk}=\frac{1}{2\pi i}\oint \frac{z^n}{p_{k+1}(z)}\,dz. \tag{75}$$

This formula corresponds to the representation (31) of the matrix p_{nv}. If the integral in (75) is evaluated by the residue theorem we obtain the formula

$$q_{nk}=\sum_{v=1}^{k}(-1)^{n-v}\frac{v^{n-1}}{(v-1)!(k-v)!}$$
$$=\frac{1}{(k-1)!}\sum_{v=1}^{k}(-1)^{n-v}\binom{k-1}{v-1}v^{n-1}.$$

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(Paper 63B1–1)

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On a Modification of Watson's Lemma

F. Oberhettinger*

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The method of steepest descents is extended to the case when a saddle point and a pole of arbitrary order are involved. An application to a problem in diffraction theory is demonstrated.

1. Introduction

The solution of numerous problems in electromagnetic and acoustic wave propagation theory is frequently given by an integral representation of the form

$$\varphi(z)=\int_C w(v)e^{-zu(v)}dv. \tag{1}$$

Here $\varphi(z)$ denotes the desired property (for instance the electromagnetic or acoustic potential). The integration in (1) is taken along a curve C in the complex v plane along which $w(v)$ and $u(v)$ are defined. These function contain generally certain parameters which depend on the physical configuration under investigation. The parameter z is usually related to the distance between some point of reference and the point of observation. A reduction of the integral (1) to an asymptotic series has to be carried out when approximate expressions for large z are desired. The frequently used method of steepest descents [3, p. 86][1] becomes invalid when one of the above mentioned parameters assumes a value for which a pole of $w(v)$ coincides with a saddle point of $u(v)$. Methods to cope with such cases have been worked out and discussed [1], [8], [9]. The former two give expansions which proceed in terms of hypergeometric functions besides a "leading term", while the latter consists of an error integral as leading term and an asymptotic series with inverse powers of z. It is shown here that the method given in [9], which involves a pole of the first order and a branch point of order 2 can be extended to the case involving a pole and a branch point of arbitrary order. The expansion for this case consists of a finite number (equal to the order of the pole) of terms involving Whittaker's functions and an asymptotic series with inverse powers of z.

2. A Modification of Watson's Lemma

It is assumed that the reduction of the contour integral (1) (for instance by the method of steepest descents) to one or several integrals of the Laplace transform type

$$f(z)=\int_0^\infty F(t)e^{-tz}dt \tag{2}$$

has already been carried out and only this kind of integrals shall be considered. A theorem commonly referred to as Watson's lemma [2, p. 218] admits the asymptotic representation of $f(z)$ for large z when the behavior of $F(t)$ for small t is known. In its general form given by Doetsch [3, p. 48] and [4] one has the

THEOREM: *Let $F(t)$ be analytic in a sector $\alpha<\arg t<\beta$ of the complex t plane with the possible exception of the origin, and let the behavior of $F(t)$ in this sector be*

(a) $|F(t)|<Ae^{c|t|}$, *for* $|t|>R$, $A, c>0$,

(b) $F(t)\approx\sum_{n=0}^{\infty}a_n t^{\alpha_n}$, $-1<\mathrm{Re}\,\alpha_0<\mathrm{Re}\,\alpha_1<\cdots$

when $t\to 0$ *in* $\alpha<\arg t<\beta$.

The asymptotic expansion for $f(z)=\int_0^\infty F(t)e^{-zt}dt$ for large z is then obtained by inserting the series (b) for $F(t)$ into (2) and integrating term by term.

$$f(z)=\int_0^\infty F(t)e^{-tz}dt\approx\sum_{n=0}^{\infty}a_n\Gamma(1+\alpha_n)z^{-\alpha_n-1} \tag{3}$$

when $z\to\infty$ in $-\frac{\pi}{2}-\beta<\arg z<\frac{\pi}{2}-\alpha$.

The function $F(t)$ in (2) is now assumed to be of the form

$$F(t)=t^\alpha g(t), \qquad \mathrm{Re}\,\alpha>-1, \tag{4}$$

where the function $g(t)$ is analytic in the vicinity of $t=0$. In this case

$$f(z)=\int_0^\infty t^\alpha g(t)e^{-tz}dt \tag{5}$$

with

$$g(t)=\sum_{n=0}^{\infty}a_n t^n, \qquad a_n=\frac{1}{n!}\left[\frac{d^n}{dt^n}\,g(t)\right]_{t=0}. \tag{6}$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

This series is convergent for $|t|<|t_0|$, where t_0 is that singular point of $g(t)$ which is closest to the origin. The other singular points are denoted by $t_1, t_2, \ldots$ such that their moduli form a nondecreasing sequence. We assume that none of the singular points is on the positive real t-axis. The asymptotic expansion of (5) then becomes by (3) and (6)

$$f(z)=\int_0^\infty t^\alpha g(t)e^{-tz}dt \approx \sum_{n=0}^{\infty} a_n \Gamma(\alpha+n+1)z^{-\alpha-n-1} \quad (7)$$

for $z\to\infty$. The range of validity for arg z depends on the behavior of $g(t)$ according to (a) and (b) of the theorem quoted before. The case in which the singular point t_0 is a pole of order m, say, is considered now. The Laurent expansion of $g(t)$ in the vicinity of t_0 is then

$$g(t)=b_{-m}(t-t_0)^{-m}+\ldots+b_{-1}(t-t_0)^{-1}+g^*(t) \quad (8)$$

where

$$g^*(t)=g(t)-b_{-m}(t-t_0)^{-m}-\ldots-b_{-1}(t-t_0)^{-1} \quad (9)$$

is analytic for $|t|<|t_1|$ and consequently can be represented by the series

$$g^*(t)=\sum_{n=0}^{\infty} c_n t^n, \qquad c_n=\frac{1}{n!}\left[\frac{d^n}{dt^n}g^*(t)\right]_{t=0}, \quad |t|<|t_1|. \quad (10)$$

It has to be remembered that in physical applications the function $g(t)$ in (4) contains generally certain parameters which depend on the configuration of the problem, and this may become such that the pole t_0 of $g(t)$ coincides with the origin $t=0$ (see section 3). In this case all the coefficients a_n tend to infinity by (6) and (8) and the asymptotic series in (7) becomes useless although the integral in (7) may exist. It is therefore necessary to obtain an expansion which for $t_0\neq 0$ represents the integral (7) asymptotically for large z and which for $t_0=0$ (if the integral exists) gives its correct value. Such an expansion can be obtained as follows: One inserts the representation (8) for $g(t)$ into (7) and gets

$$f(z)=\int_0^\infty t^\alpha g(t)e^{-tz}dt$$

$$=\sum_{k=1}^{m} b_{-k}\int_0^\infty (t-t_0)^{-k}t^\alpha e^{-tz}dt+\int_0^\infty t^\alpha g^*(t)e^{-tz}dt. \quad (11)$$

But

$$\int_0^\infty t^\alpha e^{-tz}(t-t_0)^{-k}dt=\Gamma(1+\alpha)(-t_0)^{\frac{1}{2}\alpha-\frac{1}{2}k}z^{\frac{1}{2}k-\frac{1}{2}\alpha-1}$$

$$\times e^{-\frac{1}{2}t_0 z}W_{-\frac{1}{2}k-\frac{1}{2}\alpha,\,\frac{1}{2}\alpha-\frac{1}{2}k+\frac{1}{2}}(-t_0 z), \quad (12)$$

by virtue of the integral representation for Whittaker's function [5, p. 274]. The integral

$$\int_0^\infty t^\alpha g^*(t)\,e^{-tz}dt$$

can be represented asymptotically for large z by means of the power series expansion (10) (which is valid for $|t|<|t_1|$) and Watson's lemma. One obtains then finally

$$f(z)=\int_0^\infty t^\alpha g(t)e^{-tz}dt$$

$$\approx\sum_{n=0}^{\infty} c_n\Gamma(\alpha+n+1)z^{-\alpha-n-1}+\Gamma(1+\alpha)(-t_0)^{\frac{1}{2}\alpha}z^{-\frac{1}{2}\alpha-1}$$

$$\times e^{-zt_0}\sum_{k=1}^{m} b_{-k}\left(\frac{z}{-t_0}\right)^{\frac{1}{2}k}W_{-\frac{1}{2}k-\frac{1}{2}\alpha,\,\frac{1}{2}\alpha-\frac{1}{2}k+\frac{1}{2}}(-zt_0). \quad (13)$$

The range of validity for arg z depends on the behavior of $g^*(t)$ according to the requirements (a) and (b) of the theorem quoted before. For certain combinations of α and m (13) can reduce to a simpler expression, as for instance

$$W_{k,\frac{1}{4}}(z)=2^{\frac{1}{4}-k}z^{\frac{1}{4}}D_{2k-\frac{1}{2}}[(2z)^{\frac{1}{2}}],$$

(parabolic cylinder function) (14)

$$W_{-\frac{1}{4},\frac{1}{4}}(z)=\pi^{\frac{1}{2}}z^{\frac{1}{4}}e^{\frac{1}{2}z}\,\mathrm{Erfc}\,(z^{\frac{1}{2}})=z^{\frac{1}{4}}e^{\frac{1}{2}z}\int_z^\infty t^{-\frac{1}{2}}e^{-t}dt \quad (15)$$

$$W_{-\frac{1}{2},0}(z)=-z^{-\frac{1}{2}}e^{\frac{1}{2}z}\,\mathrm{Ei}\,(-z)=z^{-\frac{1}{2}}e^{\frac{1}{2}z}\int_z^\infty t^{-1}e^{-t}dt \quad (16)$$

$$W_{0,\mu}(z)=(z/\pi)^{\frac{1}{2}}K_\mu\left(\frac{1}{2}z\right), \text{ (modified Bessel function).} \quad (17)$$

The case $\alpha=-\frac{1}{2}$, $m=1$ (branch point of order 2 and pole t_0 of the order 1) which leads to the special case (15) has been treated before [9]. For this case one obtains from (13) and (15)

$$f(z)=\int_0^\infty t^{-\frac{1}{2}}g(t)dt\approx\sum_{n=0}^{\infty}c_n\Gamma\left(n+\frac{1}{2}\right)z^{-n-\frac{1}{2}}$$
$$+b_{-1}\pi(-t_0)^{-\frac{1}{2}}e^{-zt_0}\,\mathrm{Erfc}\,[(-zt_0)^{\frac{1}{2}}]. \quad (18)$$

The coefficients c_n according to (9) and (10) are defined by

$$g(t)-b_{-1}(t-t_0)^{-1}=\sum_{n=0}^{\infty}c_nt^n, \quad (19)$$

or

$$c_n=\frac{1}{n!}\left\{\frac{d^n}{dt^n}\,[g(t)-b_{-1}(t-t_0)^{-1}]\right\}_{t=0}. \quad (20)$$

But these coefficients remain finite when $t_0\to 0$.

3. Application to the Wedge Diffraction Problem

The investigations outlined before will now be applied to a previously discussed problem. It was shown [6, 7] that the solution of the problem of the diffraction of plane waves by a wedge or corner is governed by the integral

$$I(\alpha,\psi)=\int_0^\infty t^{-\frac{1}{2}}g(t)e^{-\gamma\rho t}dt \quad (21)$$

with

$$g(t)=-(2\alpha)^{-1}e^{-\gamma\rho}\sin(\pi\psi/\alpha)(t+2)^{-\frac{1}{2}}$$
$$\cdot\{\cosh[\pi/\alpha\log(1+t+(t^2+2t)^{\frac{1}{2}})]-\cos(\pi\psi/\alpha)\}^{-1}. \quad (22)$$

Here $g(t)$ is analytic near the origin and the singularity nearest to the origin is a pole of the order one. The pole t_0 and its residue b_{-1} are found to be

$$t_0=-2\sin^2(\tfrac{1}{2}\psi) \quad (23)$$

$$b_{-1}=-\pi^{-1}2^{-\frac{1}{2}}e^{-\gamma\rho}\sin(\tfrac{1}{2}\psi). \quad (24)$$

The parameter ψ in (22) depends on the direction of the incident wave and on the direction of the point of observation such, that ψ (and consequently t_0) tends to zero when one of the boundaries of geometric optics is approached. The asymptotic expansion of the integral (21) for large $\gamma\rho$ can then be obtained by means of (18), (19), and (20):

$$I(\alpha,\psi)\approx\sum_{n=0}^{\infty}c_n\Gamma(n+\tfrac{1}{2})(\gamma\rho)^{-n-\frac{1}{2}}$$
$$-\tfrac{1}{2}e^{-\gamma\rho\cos(\frac{1}{2}\psi)}\,\mathrm{Erfc}[(2\gamma\rho)^{\frac{1}{2}}\sin\psi]\,. \quad (25)$$

The coefficients c_n are determined by (10) with

$$g^*(t)=g(t)+\pi^{-1}2^{-\frac{1}{2}}e^{-\gamma\rho}\sin(\tfrac{1}{2}\psi)(t+2\sin^2\psi/2)^{-1} \quad (26)$$

and $g(t)$ is given by (22). The function $g^*(t)$ fulfills the conditions (a) and (b) in the region $-\pi<\arg t<\pi$. The asymptotic expansion (25) holds therefore in $-3\pi/2<\arg(\gamma\rho)<3\pi/2$ and is valid also for $\psi=0$ (in which case all the c_n vanish) and assumes in this case the correct value of (21)

$$I(\alpha,0)=-\tfrac{1}{2}e^{-\gamma\rho}. \quad (27)$$

The results produced here are equivalent to those in [6,7].

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(Paper 63B1–2)

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Principal Submatrices of a Full-Rowed Non-Negative Matrix[1]

K. Goldberg

(April 2, 1959)

It is shown that one of the principal submatrices of an incidence matrix without zero rows is a permutation matrix. From this an inspection method is described for determining whether a non-negative matrix is nilpotent. Application is made to graph theory.

We begin with a necessary flock of definitions. All matrices in this paper are square matrices. The *principal submatrices* of a matrix are the matrix itself and those submatrices obtained from it by repeatedly striking out a row and the column of the same index. The *leading principal submatrices* are those obtained by striking out exactly one row and its corresponding column. The principal minors are the determinants of the principal submatrices.

A *zero vector* is one with every element equal to 0. A *full-rowed matrix* is one with no zero rows. A *non-negative matrix* is one whose elements are all non-negative real numbers. An *incidence matrix* is one whose elements are either 0 or 1. A *permutation matrix* is an incidence matrix with exactly one 1 in each row and each column. If some power of a matrix vanishes the matrix is called *nilpotent*. We shall prove:

THEOREM. *At least one of the principal submatrices of a full-rowed incidence matrix is a permutation matrix.*

COROLLARY 1. *If N is a non-negative matrix with a full-rowed principal submatrix of order m, then for some positive $r \leq m$ the trace of N^r is at least $r\eta^r$, where η is the least positive element of N.*

COROLLARY 2. *Each principal submatrix of a nilpotent non-negative matrix is nilpotent and has both zero rows and zero columns.*

Both the theorem and its first corollary hold with "row" replaced by "column". The theorem relates to similar results on systems of distinct representatives.[2]

The main effect of corollary 2 is to provide a simple algorithm for deciding whether or not a non-negative matrix is nilpotent. If it has no zero rows it is not. Otherwise the zero rows and the columns of the same index are crossed out and the process repeated on the smaller matrix. If none of the matrices examined in this process are full-rowed, then all the principal minors of the matrix are 0. The characteristic equation is thus of the form $x^n=0$,[3] and the matrix is nilpotent. R. Marimont[4] first described this process for incidence matrices.

That the principal submatrices of a nilpotent non-negative matrix are also nilpotent is known. We have included this statement in corollary 2 because a new proof follows immediately from the remaining result in this corollary.

We prove the theorem by induction. It is certainly true for a matrix of order 1. Assume that it is true for matrices of order $n-1$.

Let $A=(a_{ij})$ be a full-rowed incidence matrix of order n. If any of its leading principal submatrices is full-rowed then we are finished because "principal submatrix" is a hereditary property. Therefore, we assume that each of the leading principal submatrices of A has at least one zero row. We shall prove that then A is a permutation matrix.

Let A_s denote the leading principal submatrix of A obtained by striking out the row and column of index s. We may assume that there is an index i_s such that

$$a_{i_s j}=0 \quad \text{for } j=1,2,\ldots,s-1,s+1,\ldots,n. \tag{2.1}$$

Since no row of A is a zero row this implies that

$$a_{i_s s}=1. \tag{2.2}$$

If another leading principal submatrix A_t also had as one of its zero rows a row corresponding to the index i_s we would have $a_{i_s t}=1$ with $t \neq s$. This

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would contradict (2.1). Therefore, to each s there corresponds a unique i_s which satisfies (2.1) and (2.2). It follows that A is a permutation matrix, and the theorem is proved.

For any non-negative matrix N let $\overline{N}$ denote the incidence matrix of the same order as N, with 1 in every position that N has a positive element and 0 in every position that N has 0. If η is the least positive element in N then $N-\eta\overline{N}$ is a non-negative matrix. Thus, if tr X denotes the trace of X,

$$\text{tr } N^r \geq \eta^r \text{ tr } \overline{N}^r$$

for every non-negative integer r.

Furthermore, let $N=(n_{ij})$ be of order n, let $S=\{1,2, \ldots, n\}$, and let S_0 be any subset of S. Let M be the principal submatrix of N with row and column indices in S_0. Then for any positive integer r we have

$$\text{tr } N^r=\sum_{i_j \in S} n_{i_1 i_2} n_{i_2 i_3} \cdots n_{i_r i_1}$$

$$\geq \sum_{i_j \in S_0} n_{i_1 i_2} n_{i_2 i_3} \cdots n_{i_r i_1}=\text{tr } M^r.$$

Now suppose M is a full-rowed principal submatrix of N of order m. Then $\overline{M}$ is a full-rowed incidence matrix, and so has a principal submatrix P of order $s \leq m$ which is a permutation matrix. Let $r \leq S$ be any of the cycle lengths of the permutation corresponding to P. Then

$$\text{tr } P^r \geq r.$$

Since P is a principal submatrix of $\overline{M}$, the incidence matrix corresponding to M, and M is a principal submatrix of N, we have

$$\text{tr } N^r \geq \text{tr } M^r \geq \eta^r \text{ tr } \overline{M}^r \geq \eta^r \text{ tr } P^r \geq r\eta^r$$

with $r \leq s \leq m$. This proves corollary 1.

Corollary 2 follows directly from corollary 1 and the fact that a matrix is nilpotent (if and) only if the traces of its positive powers are all 0.

Our theorem can also be stated as a result on directed graphs. A directed graph G is a set of points $P=\{p,q, \ldots\}$ and a subset $P^*=\{(p,q), \ldots\}$ of the Cartesian product $P \times P$. The ordered pair (p,q) may be thought of as a directed line from p, called the starting point, to q

We say that G contains a *simple loop* if there is a subset $P'=\{p'_1,p'_2, \ldots,p'_n\}$ of P such that

$$P^* \cap (P' \times P')=\{(p'_1,p'_2),(p'_2,p'_3), \ldots,$$

$$(p'_{n-1},p'_n), (p'_n,p'_1)\}.$$

That is to say, the points of P' have these lines between them, *and no others*.

Our theorem is equivalent to the following statement: *If every point of a directed graph is a starting point then the graph contains a simple loop.*

As so stated, this theorem is essentially one proved by David Rosenblatt.[5]

[5] D. Rosenblatt, On the graphs and asymptotic forms of finite Boolean relation matrices and stochastic matrices, Naval Res. Log. Quart. **4**, 151 (1957).

WASHINGTON, D.C. (Paper 63B1–3)

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Zeros of Certain Polynomials

A. J. Goldman

(March 2, 1959)

Let P be a real parameter. It is proved that all roots of $z^{n+1}-z^n+P=0$ lie in the open unit disk, if and only if $0<P<2\sin\pi/(4n+2)$.

1. *Introduction.* Let n be a fixed non-negative integer. The following problem will be solved:

PROBLEM. *Find the set S_n of all values of the real parameter P for which all roots of*

$$z^{n+1}-z^n+P=0 \tag{1}$$

lie in the open unit disk $|z|<1$.

In view of the substantial literature relating the coefficients of a polynomial to the locations of its zeros,[1] one would expect that some standard algorithm could be applied to so specific a problem. This seems to be only partly true; the appropriate algorithm is that given by the Schur-Cohen Criterion,[2] which yields only an implicit characterization of S_n as the solution-set of a system of $n+1$ polynomial inequalities in P. It appears nontrivial to derive from this an *explicit* characterization of S_n, and so we give instead an elementary self-contained solution. The result is the

THEOREM. $S_n=\{P|\ 0<P<2\sin\pi/(4n+2)\}$.

The problem arose in connection with the generating function [3]

$$F(s)=\frac{p^n s^n(1-ps)}{1-s+(1-p)p^n s^{n+1}}$$

of the recurrence times for runs of n successes in a sequence of Bernoulli trials with "success probability" p. Rigorous justification of the usual probability theory manipulations of the power series for $F(s)$ is easy if $F(s)$ has no singularities for $|s|\leq 1$, and this is in fact true and is equivalent (upon setting $z=1/s$) to the assertion that all roots of $z^{n+1}-z^n+(1-p)p^n=0$ lie in $|z|<1$ for $0<p<1$. The last statement shows that S_n *includes* the interval $\{P|\ 0<P<n^n/(n+1)^{n+1}\}$, and it was natural to inquire whether this expression gave S_n exactly. It follows from our theorem that S_n is larger than this interval for all $n>0$, and is approximately $\frac{1}{2}\pi e\approx 4.27$ times as long for large n.

[1] Marden, The geometry of the zeros of a polynomial in a complex variable, Am. Math. Soc. Math. Survey No. 3 (New York, N.Y., 1949).
[2] Ibid, p. 152.
[3] Feller, Probability theory and its applications, p. 265 (John Wiley & Sons, Inc., New York, N.Y., 1950

2. *Solution.* In this section x, r, and θ denote real variables obeying $r>0$, $0\leq\theta<2\pi$. Since the theorem stated above is obviously true for $n=0$, it is assumed that $n>0$ in what follows. It is convenient to define

$$f(\theta)=\sin^n n\theta\sin\theta/\sin^{n+1}(n+1)\theta,$$
$$A=\{x^n-x^{n+1}|\ |x|\geq 1\},$$
$$B=\{\theta|\ 0<\theta<\pi,\ \sin n\theta\geq\sin(n+1)\theta>0\},$$
$$C=\{f(\theta)|\ \theta \text{ in } B\}.$$

We can omit the easy proof of

LEMMA 1. $A=\{P|\ P\leq 0 \text{ or } P\geq 2\}$ *if n is even,*
$A=\{P|\ P\leq 0\}$ *if n is odd.*

LEMMA 2. $C=\{P|\ P\geq 2\sin\pi/(4n+2)\}$.

PROOF. (a) From the formula

$$f'(\theta)=\frac{\sin^{n-1}n\theta[(n\sin\theta-\sin n\theta)^2+2n\sin\theta\sin n\theta(1-\cos(n+1)\theta)]}{\sin^{n+2}(n+1)\theta}$$

we conclude that $f(\theta)$ is increasing on each subinterval of B.

(b) Suppose $\sin(n+1)\theta^*=0$ at a left endpoint θ^* of some maximal subinterval of B. If $\sin n\theta^*\neq 0$, then $\sin\theta^*\neq 0$ and so $f(\theta^*+)=(+\infty)$, contradicting (a). If $\sin n\theta^*=0$, then $\theta^*=0$, contradicting the requirement that

$$\sin n(\theta^*+\delta)\geq\sin(n+1)(\theta^*+\delta)$$

for all sufficiently small $\delta>0$. Thus the supposition is untenable.

(c) We next apply the identity

$$\sin n\theta-\sin(n+1)\theta=-2\sin\frac{1}{2}\theta\cos\frac{1}{2}(2n+1)\theta$$

to obtain

$$B=\{\theta|\ 0<\theta<\pi,\ \sin(n+1)\theta>0,\ \cos\frac{1}{2}(2n+1)\theta\leq 0\}.$$

Consider now any left endpoint θ^* of a maximal subinterval of B. According to (b), we must have

$$\sin(n\theta^*)=\sin(n+1)\theta^*, \tag{2}$$

so that

$$f(\theta^*)=\sin\theta^*/\sin(n+1)\theta^*. \tag{3}$$

In fact, the conditions determining such an endpoint are, in addition to (2) and $0\leq\theta^*<\pi$, that

$$\sin(n+1)\theta^*>0 \text{ and } \sin n(\theta^*+\delta)>\sin(n+1)(\theta^*+\delta),$$

for all sufficiently small $\delta>0$. Equivalently, θ^* is such an endpoint if and only if

$$\cos\frac{1}{2}(2n+1)\theta^*=0, \tag{4}$$

$$\cos\frac{1}{2}(2n+1)(\theta^*+\delta)<0, \qquad (\delta \text{ as above}) \tag{5}$$

$$\sin(n+1)\theta^*>0, \tag{6}$$

$$0\leq\theta^*<\pi. \tag{7}$$

The points obeying (4), (5), and (7) are precisely the points

$$\theta_j=(4j+1)\pi/(2n+1) \qquad (0\leq 2j<2n);$$

these points also satisfy (6), since

$$\sin(n+1)\theta_j=\cos(4j+1)\pi/(4n+2) \qquad (4j+1<4n+2), \tag{8}$$

and so the θ_j's are precisely the left endpoints of the maximal subintervals of B. From (3) and (8) we have

$$f(\theta_j)=2\sin(4j+1)\pi/(4n+2);$$

together with (a), this shows that $f(\theta)$ reaches its minimum on B at $\theta=\theta_0=\pi/(2n+1)$, this minimum being

$$\min C=2\sin\pi/(4n+2).$$

(d) Finally, we seek the right endpoint $\bar{\theta}$ of that maximal subinterval of B of which θ_0 is the left endpoint. After θ_0, $\cos\frac{1}{2}(2n+1)\theta$ first changes sign at $3\pi/(2n+1)$, but $\sin(n+1)\theta$ changes sign earlier, at $\pi/(n+1)$. Thus $\bar{\theta}=\pi/(n+1)$, and so $f(\bar{\theta}-)=(+\infty)$. This fact, together with (a) and the results of (c), completes the proof.

Our final lemma gives the motivation for lemmas 1 and 2; the three lemmas together immediately imply the theorem stated in the introduction.

LEMMA 3. *S_n is the complement of $A\cup C$.*

PROOF. (a) Clearly eq (1) has a *real* root outside the disk $|z|<1$ if and only if P is in A.

(b) Next we observe that $z=r\exp(i\theta)$ is a nonreal root of eq (1) if and only if

$$r^n\sin n\theta-r^{n+1}\sin(n+1)\theta=0, \tag{9}$$

$$\sin\theta\neq 0, \tag{10}$$

$$r^n\cos n\theta-r^{n+1}\cos(n+1)\theta=P. \tag{11}$$

Now (9) and (10) are equivalent to

$$\sin n\theta/\sin(n+1)\theta=r, \tag{12}$$

and (11) and (12) are equivalent to

$$f(\theta)=P. \tag{13}$$

Thus eq (1) has a nonreal root outside the disk $|z|<1$ if and only if P lies in the set

$$\{f(\theta)|\ \sin n\theta/\sin(n+1)\theta\geq 1\},$$

which (by considering the change $\theta\to 2\pi-\theta$) is readily seen to be identical with the set

$$\{f(\theta)|\ \sin n\theta\geq\sin(n+1)\theta>0\}.$$

(c) By (a) and (b), S_n is the complement of

$$A\cup\{f(\theta)|\ \sin n\theta\geq\sin(n+1)\theta>0\}.$$

By lemma 1, this union is identical with

$$A\cup\{f(\theta)|\ \sin n\theta\geq\sin(n+1)\theta>0,\quad f(\theta)>0\},$$

which (by the form of $f(\theta)$) is just $A\cup C$.

WASHINGTON, D.C. (Paper 63B1–4)

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Tables of Transport Integrals: A Supplement

William M. Rogers,[1] William J. Hall, and Robert L. Powell

(April 23, 1959)

Tables of values of the transport integrals, defined by

$$J_n(x) \equiv \int_0^x \frac{e^z z^n dz}{(e^z-1)^2},$$

were previously published in NBS Circular 595 for values of n from 2 through 17. In this paper, values are given for the integrals where n is 18 and 20 and where x ranges from 0.2 to 50.0 in steps of 0.2.

Soon after the publication of the Tables of Transport Integrals [2] an unanticipated need arose for tables of the integrals for n equal to 18 and 20. The integrals with even values of n are useful in the problem of inverting heat capacity results to obtain lattice frequency spectra for solids.[3] The technique is similar to the method of moments developed by Montroll.[4] The integrals with odd values of n are not used in this method, and since there was no other need for the integrals with n above 17, the integrals with $n=19$ were not calculated. The additional values were calculated on the IBM 650 computer using the already existent programs and auxiliary numbers. The computed values of the integrals are presented in the following tables.

The mathematical formulation, analyses of errors, and tables of auxiliary numbers were discussed in the previous publication. For x below 10, interpolation should be performed by the technique of using an interpolating polynomial in a table of divided differences of the logarithms of both the arguments and the integral values. For x between 0 and 4, a fifth-degree polynomial should be used; between 4 and 10, a third-degree polynomial. For x above 10, straightforward Lagrangian interpolation should be performed. For x between 10 and 20, a 4-point formula should be used; between 20 and 35, a 3-point; and above 35, a 2-point.

[1] Present address: Headquarters, Strategic Air Command, Offutt Air Force Base, Nebr.
[2] William M. Rogers and Robert L. Powell, Tables of transport integrals, NBS Circ. 595.
[3] J. A. Morrison, private communication.
[4] E. W. Montroll, J. Chem. Phys. **11**, 481 (1943).

(Tables follow)

TRANSPORT INTEGRALS : $J_n(x) \equiv \int_0^x \frac{e^z z^n}{(e^z-1)^2} dz$; $n = 18, 20$

x	1/x	$J_{18}(x)$		$J_{20}(x)$	
		COEF.	EXP.	COEF.	EXP.
.2	5.0000	7.68716	-14	2.75110	-15
.4	2.5000	9.98611	- 9	1.42940	- 9
.6	1.6667	9.69389	- 6	3.12152	- 6
.8	1.2500	1.26317	- 3	7.22945	- 4
1.0	1.0000	5.46290	- 2	4.88381	- 2
1.2	.8333	1.17362	0	1.51031	0
1.4	.7143	1.55318	1	2.71939	1
1.6	.6250	1.43986	2	3.29112	2
1.8	.5556	1.01589	3	2.93725	3
2.0	.5000	5.77334	3	2.05955	4
2.2	.4545	2.75197	4	1.18710	5
2.4	.4167	1.13372	5	5.81587	5
2.6	.3846	4.12961	5	2.48433	6
2.8	.3571	1.35385	6	9.43812	6
3.0	.3333	4.05157	6	3.23956	7
3.2	.3125	1.11950	7	1.01753	8
3.4	.2941	2.88300	7	2.95532	8
3.6	.2778	6.97382	7	8.00642	8
3.8	.2632	1.59495	8	2.03807	9
4.0	.2500	3.46812	8	4.90503	9
4.2	.2381	7.20417	8	1.12206	10
4.4	.2273	1.43552	9	2.45095	10
4.6	.2174	2.75377	9	5.13255	10
4.8	.2083	5.10162	9	1.03403	11
5.0	.2000	9.15275	9	2.01034	11
5.2	.1923	1.59414	10	3.78208	11
5.4	.1852	2.70137	10	6.90194	11
5.6	.1786	4.46243	10	1.22443	12
5.8	.1724	7.19868	10	2.11576	12
6.0	.1667	1.13584	11	3.56723	12
6.2	.1613	1.75544	11	5.87786	12
6.4	.1563	2.66088	11	9.47887	12
6.6	.1515	3.96048	11	1.49801	13
6.8	.1471	5.79463	11	2.32279	13
7.0	.1429	8.34235	11	3.53773	13
7.2	.1389	1.18285	12	5.29775	13
7.4	.1351	1.65316	12	7.80755	13
7.6	.1316	2.27919	12	1.13335	14
7.8	.1282	3.10193	12	1.62174	14
8.0	.1250	4.17019	12	2.28921	14

TRANSPORT INTEGRALS: $J_n(x) \equiv \int_0^x \frac{e^z z^n}{(e^z-1)^2}\, dz$; $n = 18, 20$ (cont'd.)

x	1/x	$J_{18}(x)$		$J_{20}(x)$	
		COEF.	EXP.	COEF.	EXP.
8.2	.1220	5.54145	12	3.18984	14
8.4	.1190	7.28248	12	4.39041	14
8.6	.1163	9.47008	12	5.97241	14
8.8	.1136	1.21916	13	8.03411	14
9.0	.1111	1.55454	13	1.06928	15
9.2	.1087	1.96411	13	1.40870	15
9.4	.1064	2.45996	13	1.83786	15
9.6	.1042	3.05533	13	2.37590	15
9.8	.1020	3.76441	13	3.04320	15
10.0	.1000	4.60253	13	3.86510	15
10.2	.0980	5.58606	13	4.86890	15
10.4	.0962	6.73205	13	6.08530	15
10.6	.0943	8.05832	13	7.54830	15
10.8	.0926	9.58320	13	9.29500	15
11.0	.0909	1.13254	14	1.13656	16
11.2	.0893	1.33040	14	1.38045	16
11.4	.0877	1.55380	14	1.66582	16
11.6	.0862	1.80461	14	1.99760	16
11.8	.0847	2.08468	14	2.38115	16
12.0	.0833	2.39576	14	2.82181	16
12.2	.0820	2.73954	14	3.32528	16
12.4	.0806	3.11761	14	3.89742	16
12.6	.0794	3.53141	14	4.54416	16
12.8	.0781	3.98224	14	5.27146	16
13.0	.0769	4.47120	14	6.08533	16
13.2	.0758	4.99927	14	6.99174	16
13.4	.0746	5.56715	14	7.99644	16
13.6	.0735	6.17539	14	9.10518	16
13.8	.0725	6.82427	14	1.03233	17
14.0	.0714	7.51388	14	1.16559	17
14.2	.0704	8.24402	14	1.31077	17
14.4	.0694	9.01431	14	1.46831	17
14.6	.0685	9.82409	14	1.63858	17
14.8	.0676	1.06725	15	1.82193	17
15.0	.0667	1.15583	15	2.01862	17
15.2	.0658	1.24803	15	2.22887	17
15.4	.0649	1.34370	15	2.45283	17
15.6	.0641	1.44264	15	2.69057	17
15.8	.0633	1.54468	15	2.94210	17
16.0	.0625	1.64960	15	3.20737	17

TRANSPORT INTEGRALS: $J_n(x) \equiv \int_0^x \frac{e^z z^n}{(e^z-1)^2}\,dz$; $n = 18, 20$ (cont'd.)

x	1/x	$J_{18}(x)$		$J_{20}(x)$	
		COEF.	EXP.	COEF.	EXP.
16.2	.0617	1.75718	15	3.48623	17
16.4	.0610	1.86717	15	3.77849	17
16.6	.0602	1.97933	15	4.08386	17
16.8	.0595	2.09340	15	4.40200	17
17.0	.0588	2.20911	15	4.73249	17
17.2	.0581	2.32619	15	5.07485	17
17.4	.0575	2.44436	15	5.42855	17
17.6	.0568	2.56336	15	5.79297	17
17.8	.0562	2.68289	15	6.16747	17
18.0	.0556	2.80270	15	6.55134	17
18.2	.0549	2.92250	15	6.94383	17
18.4	.0543	3.04204	15	7.34417	17
18.6	.0538	3.16107	15	7.75153	17
18.8	.0532	3.27933	15	8.16507	17
19.0	.0526	3.39658	15	8.58393	17
19.2	.0521	3.51262	15	9.00721	17
19.4	.0515	3.62720	15	9.43404	17
19.6	.0510	3.74015	15	9.86352	17
19.8	.0505	3.85127	15	1.02948	18
20.0	.0500	3.96039	15	1.07269	18
20.2	.0495	4.06735	15	1.11590	18
20.4	.0490	4.17201	15	1.15902	18
20.6	.0485	4.27423	15	1.20198	18
20.8	.0481	4.37390	15	1.24469	18
21.0	.0476	4.47093	15	1.28707	18
21.2	.0472	4.56522	15	1.32905	18
21.4	.0467	4.65671	15	1.37056	18
21.6	.0463	4.74534	15	1.41152	18
21.8	.0459	4.83107	15	1.45189	18
22.0	.0455	4.91386	15	1.49159	18
22.2	.0450	4.99369	15	1.53058	18
22.4	.0446	5.07056	15	1.56881	18
22.6	.0442	5.14447	15	1.60622	18
22.8	.0439	5.21543	15	1.64278	18
23.0	.0435	5.28346	15	1.67846	18
23.2	.0431	5.34860	15	1.71322	18
23.4	.0427	5.41089	15	1.74703	18
23.6	.0424	5.47037	15	1.77987	18
23.8	.0420	5.52709	15	1.81173	18
24.0	.0417	5.58111	15	1.84259	18

'd.) TRANSPORT INTEGRALS: $J_n(x) \equiv \int_0^x \frac{e^z z^n}{(e^z-1)^2}\, dz$; $n = 18, 20$ (cont'd.)

x	1/x	$J_{18}(x)$		$J_{20}(x)$	
		COEF.	EXP.	COEF.	EXP.
24.2	.0413	5.63250	15	1.87243	18
24.4	.0410	5.68132	15	1.90126	18
24.6	.0407	5.72765	15	1.92907	18
24.8	.0403	5.77155	15	1.95585	18
25.0	.0400	5.81312	15	1.98162	18
25.2	.0397	5.85242	15	2.00638	18
25.4	.0394	5.88954	15	2.03014	18
25.6	.0391	5.92455	15	2.05290	18
25.8	.0388	5.95755	15	2.07470	18
26.0	.0385	5.98862	15	2.09553	18
26.2	.0382	6.01783	15	2.11543	18
26.4	.0379	6.04526	15	2.13441	18
26.6	.0376	6.07101	15	2.15249	18
26.8	.0373	6.09514	15	2.16969	18
27.0	.0370	6.11774	15	2.18604	18
27.2	.0368	6.13889	15	2.20157	18
27.4	.0365	6.15865	15	2.21629	18
27.6	.0362	6.17710	15	2.23025	18
27.8	.0360	6.19431	15	2.24345	18
28.0	.0357	6.21034	15	2.25593	18
28.2	.0355	6.22528	15	2.26772	18
28.4	.0352	6.23917	15	2.27885	18
28.6	.0350	6.25208	15	2.28933	18
28.8	.0347	6.26407	15	2.29921	18
29.0	.0345	6.27519	15	2.30849	18
29.2	.0342	6.28550	15	2.31722	18
29.4	.0340	6.29505	15	2.32542	18
29.6	.0338	6.30388	15	2.33311	18
29.8	.0336	6.31205	15	2.34031	18
30.0	.0333	6.31960	15	2.34706	18
30.2	.0331	6.32657	15	2.35337	18
30.4	.0329	6.33299	15	2.35927	18
30.6	.0327	6.33892	15	2.36478	18
30.8	.0325	6.34437	15	2.36992	18
31.0	.0323	6.34939	15	2.37472	18
31.2	.0321	6.35401	15	2.37918	18
31.4	.0318	6.35825	15	2.38333	18
31.6	.0316	6.36214	15	2.38720	18
31.8	.0314	6.36572	15	2.39079	18
32.0	.0313	6.36899	15	2.39412	18

TRANSPORT INTEGRALS: $J_n(x) \equiv \int_0^x \frac{e^z z^n}{(e^z-1)^2}\,dz$; $n = 18, 20$ (cont'd.)

x	1/x	$J_{18}(x)$		$J_{20}(x)$	
		COEF.	EXP.	COEF.	EXP.
32.2	.0311	6.37199	15	2.39722	18
32.4	.0309	6.37474	15	2.40008	18
32.6	.0307	6.37726	15	2.40274	18
32.8	.0305	6.37956	15	2.40520	18
33.0	.0303	6.38166	15	2.40747	18
33.2	.0301	6.38358	15	2.40957	18
33.4	.0299	6.38533	15	2.41151	18
33.6	.0298	6.38692	15	2.41330	18
33.8	.0296	6.38838	15	2.41496	18
34.0	.0294	6.38970	15	2.41648	18
34.2	.0292	6.39091	15	2.41788	18
34.4	.0291	6.39200	15	2.41917	18
34.6	.0289	6.39300	15	2.42036	18
34.8	.0287	6.39391	15	2.42145	18
35.0	.0286	6.39473	15	2.42245	18
35.2	.0284	6.39548	15	2.42337	18
35.4	.0282	6.39615	15	2.42421	18
35.6	.0281	6.39677	15	2.42499	18
35.8	.0279	6.39732	15	2.42570	18
36.0	.0278	6.39783	15	2.42634	18
36.2	.0276	6.39828	15	2.42694	18
36.4	.0275	6.39869	15	2.42748	18
36.6	.0273	6.39907	15	2.42798	18
36.8	.0272	6.39940	15	2.42843	18
37.0	.0270	6.39971	15	2.42884	18
37.2	.0269	6.39998	15	2.42922	18
37.4	.0267	6.40023	15	2.42956	18
37.6	.0266	6.40045	15	2.42988	18
37.8	.0265	6.40065	15	2.43016	18
38.0	.0263	6.40083	15	2.43042	18
38.2	.0262	6.40099	15	2.43066	18
38.4	.0260	6.40114	15	2.43087	18
38.6	.0259	6.40127	15	2.43107	18
38.8	.0258	6.40139	15	2.43124	16
39.0	.0256	6.40150	15	2.43140	18
39.2	.0255	6.40159	15	2.43155	18
39.4	.0254	6.40168	15	2.43168	18
39.6	.0253	6.40175	15	2.43180	18
39.8	.0251	6.40182	15	2.43191	18
40.0	.0250	6.40188	15	2.43201	18

TRANSPORT INTEGRALS: $J_n(x) \equiv \int_0^x \frac{e^z z^n}{(e^z-1)^2}\, dz$; $n = 18, 20$ (cont'd.)

x	1/x	$J_{18}(x)$		$J_{20}(x)$	
		COEF.	EXP.	COEF.	EXP.
40.2	.0249	6.40194	15	2.43210	18
40.4	.0248	6.40199	15	2.43218	18
40.6	.0246	6.40203	15	2.43225	18
40.8	.0245	6.40207	15	2.43232	18
41.0	.0244	6.40211	15	2.43238	18
41.2	.0243	6.40214	15	2.43243	18
41.4	.0242	6.40217	15	2.43248	18
41.6	.0240	6.40219	15	2.43252	18
41.8	.0239	6.40222	15	2.43256	18
42.0	.0238	6.40224	15	2.43260	18
42.2	.0237	6.40225	15	2.43263	18
42.4	.0236	6.40227	15	2.43266	18
42.6	.0235	6.40228	15	2.43268	18
42.8	.0234	6.40230	15	2.43270	18
43.0	.0233	6.40231	15	2.43273	18
43.2	.0231	6.40232	15	2.43274	18
43.4	.0230	6.40233	15	2.43276	18
43.6	.0229	6.40234	15	2.43278	18
43.8	.0228	6.40234	15	2.43279	18
44.0	.0227	6.40235	15	2.43280	18
44.2	.0226	6.40235	15	2.43281	18
44.4	.0225	6.40236	15	2.43282	18
44.6	.0224	6.40236	15	2.43283	18
44.8	.0223	6.40237	15	2.43284	18
45.0	.0222	6.40237	15	2.43285	18
45.2	.0221	6.40237	15	2.43285	18
45.4	.0220	6.40238	15	2.43286	18
45.6	.0219	6.40238	15	2.43286	18
45.8	.0218	6.40238	15	2.43287	18
46.0	.0217	6.40238	15	2.43287	18
46.2	.0216	6.40239	15	2.43288	18
46.4	.0216	6.40239	15	2.43288	18
46.6	.0215	6.40239	15	2.43288	18
46.8	.0214	6.40239	15	2.43288	18
47.0	.0213	6.40239	15	2.43289	18
47.2	.0212	6.40239	15	2.43289	18
47.4	.0211	6.40239	15	2.43289	18
47.6	.0210	6.40239	15	2.43289	18
47.8	.0209	6.40239	15	2.43289	18
48.0	.0208	6.40239	15	2.43289	18

TRANSPORT INTEGRALS: $J_n(x) \equiv \int_0^x \frac{e^z z^n}{(e^z-1)^2}\, dz$; $n = 18, 20$ (cont'd.)

x	1/x	$J_{18}(x)$		$J_{20}(x)$	
		COEF.	EXP.	COEF.	EXP.
48.2	.0207	6.40239	15	2.43290	18
48.4	.0207	6.40239	15	2.43290	18
48.6	.0206	6.40240	15	2.43290	18
48.8	.0205	6.40240	15	2.43290	18
49.0	.0204	6.40240	15	2.43290	18
49.2	.0203	6.40240	15	2.43290	18
49.4	.0202	6.40240	15	2.43290	18
49.6	.0202	6.40240	15	2.43290	18
49.8	.0201	6.40240	15	2.43290	18
50.0	.0200	6.40240	15	2.43290	18

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Lens Design: A New Approach[1]

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This paper describes a new method of defining the total aberrations of an optical system and its application to lens design. A system of partial differential equations defining optical image formation is written in the form of 4 by 4 matrices and is applied to the derivation of lens aberrations. It is shown that the form of the matrices facilitates the computation of these aberrations and that equations of condition on the lens parameters can be derived. The use of this method is illustrated through its application to a simple, well-known problem.

1. Introduction

In the course of designing a lens, one invariably computes aberration functions whose magnitudes reflect qualitatively and quantitatively the performance of an optical system. Unfortunately these quantities are rather difficult to calculate, usually requiring a long program of ray tracing, and, more importantly, are not readily correlated with the lens parameters. Usually a lens designer must rely on his experience when determining whether a change in a certain curvature, thickness, or index of refraction will affect a given aberration.

In this paper a method of attack will be described which will at least mitigate these difficulties. The application of this method results in the definition of a set of aberrations together with a systematic scheme for their computation expressed as equations in which the system parameters appear. Yet the method is sufficiently flexible to meet the special requirements of any lens. For example, one system of aberrations can be defined for use with a copy lens and a second system can be constructed for application to an afocal system.

Consider an optical system consisting of a system of lenses and fixed reference planes in object and image space which are not necessarily object and image planes. Right-handed coordinate systems are chosen in object and image space, the coordinates of which are designated x, y, z, and x', y', z', respectively. We take the positive z and z' axes in the general direction of ray propagation and x-y plane and x'-y' plane coincident with the object and image reference planes. Let the reduced direction cosines of a ray in object and image space be ξ, η, ζ and ξ', η', ζ', where $\xi^2+{}^2\eta+\zeta^2=N^2$, $\xi'^2+\eta'^2+\zeta'^2=N'^2$, N and N' being the indices of refraction of object and image space.

Associated with each ray passing through the optical system are two sets of four independent coordinates, x, y, ξ, η, and x', y', ξ', η'; for a given optical system either set completely determines a ray. In an obvious way a given optical system can be said to define x', y', ξ', and η' as functions of x, y, ξ, and η. In what follows we tacitly assume that in any optical system these four functions are continuous and possess continuous first-partial derivatives.

Suppose every ray emanating from an object point converges to a single point on the image plane. Then x' and y', considered as functions of the object space variables, are unaffected by a change of ξ and η. Thus x' and y' are independent of ξ and η and

$$x'_\xi=x'_\eta=y'_\xi=y'_\eta=0 \tag{1}$$

where the subscripts denote partial derivatives. The converse is also true; if there exists a pair of values for x and y for which (1) is satisfied, then all rays emanating from the point (x, y) in object space converge to a single point (x', y') in image space. Thus the four equations of (1)

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provide a necessary and sufficient condition for a point to be imaged perfectly. By a similar argument it can be seen that in a region where (1) is satisfied a necessary and sufficient condition for a lens to be free from distortion is that

$$x'_x = y'_y = m, \qquad x'_y = y'_x = 0, \tag{2}$$

where m is a constant and represents the magnification of the system.

These two sets of equations provide conditions assuring perfect, distortion-free image formation. By the same token they may be taken as measures of the failure of an optical system to achieve perfection; that is to say, they are measures of the aberrations of the system. What follows is an elaboration of this basic idea.

2. The Fundamental Optical Invariant

The functional relationship described in the preceding section is a general transformation of which optical image formation is a special case. It is desirable therefore to impose restrictions on the functions; to exclude from our attention those relationships that do not apply to optical systems. There are several ways in which this can be done. For the purposes of this study the most convenient method is through the use of Herzberger's fundamental optical invariant. This leads to a system of partial differential equations which must be satisfied by the functions x', y', ξ', and η' if they are to describe the image formation of an optical system. Following Herzberger [1] these are written,

$$\left.\begin{aligned}
x'_x\xi'_y + y'_x\eta'_y - x'_y\xi'_x - y'_y\eta'_x &= 0\\
x'_x\xi'_\xi + y'_x\eta'_\xi - x'_\xi\xi'_x - y'_\xi\eta'_x &= 1\\
x'_x\xi'_\eta + y'_x\eta'_\eta - x'_\eta\xi'_x - y'_\eta\eta'_x &= 0\\
x'_y\xi'_\xi + y'_y\eta'_\xi - x'_\xi\xi'_y - y'_\xi\eta'_y &= 0\\
x'_y\xi'_\eta + y'_y\eta'_\eta - x'_\eta\xi'_y - y'_\eta\eta'_y &= 1\\
x'_\xi\xi'_\eta + y'_\xi\eta'_\eta - x'_\eta\xi'_\xi - y'_\eta\eta'_\xi &= 0.
\end{aligned}\right\} \tag{3}$$

It is particularly convenient to express these in matrix form as

$$\overline{X}JX = J, \tag{4}$$

where

$$X = \begin{pmatrix} x'_x & x'_y & x'_\xi & x'_\eta \\ y'_x & y'_y & y'_\xi & y'_\eta \\ \xi'_x & \xi'_y & \xi'_\xi & \xi'_\eta \\ \eta'_x & \eta'_y & \eta'_\xi & \eta \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

and $\overline{X}$ represents the transpose of X.

It is important to note that X is a Jacobian matrix. An interesting corollary to (4) is that

$$|X| = 1, \tag{5}$$

or, in words, the Jacobian of the transformation representing optical image formation is equal to unity.

It is interesting to note that this corollary is equivalent to a theorem proven by Straubel [2] as long ago as 1902. He applied this to several problems including photometry. More recently Toraldo di Francia [3] used this theorem in describing a generalization of geometric optics.

The X-matrix is particularly useful. First of all its elements consist of the partial derivatives of image space variables with respect to object space variables and therefore can be said to represent the geometrical aberrations of an optical system. Moreover, it appears as an indeterminate in the equations derived from the fundamental optical invariant. Finally it can be decomposed into a product of X-matrices each representing a component of the lens. Thus, if we consider an optical system consisting of two lenses arranged so that the image plane of the first coincides with the object plane of the second then

$$X=X_2X_1 \tag{6}$$

where X_1 is the X-matrix of the first lens, X_2 of the second and X that of the composite optical system.

The proof of (6) follows at once from the properties of Jacobian matrices and the rules for differentiating a function of a function. Let the first lens be defined by the four functions $\bar{x}$, $\bar{y}$, $\bar{\xi}$, and $\bar{\eta}$ of x, y, ξ, and η and let the second be defined by x', y', ξ', and η' of $\bar{x}$, $\bar{y}$, ζ, and $\bar{\eta}$. The composite lens is then defined by x', y', ξ', and η' as functions of x, y, ξ, η. Then differentiating

$$x'_x=x'_{\bar{x}}\bar{x}_x+y'_{\bar{y}}\bar{y}_x+x'_{\bar{\xi}}\xi_x+x'_{\bar{\eta}}\bar{\eta}_x$$

$$y'_x=y'_{\bar{x}}\bar{x}_x+y'_{\bar{y}}\bar{y}_x+y'_{\bar{\xi}}\xi_x+y'_{\bar{\eta}}\bar{\eta}_x, \text{ etc.}$$

Then

$$X=\begin{pmatrix} x'_x & x'_y & x'_\xi & x'_\eta \\ y'_x & y'_y & y'_\xi & y'_\eta \\ \xi'_x & \xi'_y & \xi'_\xi & \xi'_\eta \\ \eta'_x & \eta'_y & \eta'_\xi & \eta'_\eta \end{pmatrix}=\begin{pmatrix} x'_{\bar{x}} & x'_{\bar{y}} & x'_{\bar{\xi}} & x'_{\bar{\eta}} \\ y'_{\bar{x}} & y'_{\bar{y}} & y'_{\bar{\xi}} & y'_{\bar{\eta}} \\ \xi'_{\bar{x}} & \xi'_{\bar{y}} & \xi'_{\bar{\xi}} & \xi'_{\bar{\eta}} \\ \eta'_{\bar{x}} & \eta'_{\bar{y}} & \eta'_{\bar{\xi}} & \eta'_{\bar{\eta}} \end{pmatrix}\begin{pmatrix} \bar{x}_x & \bar{x}_y & \bar{y}_\xi & \bar{x}_\eta \\ \bar{y}_x & \bar{y}_y & \bar{y}_\xi & \bar{y}_\eta \\ \bar{\xi}_x & \bar{\xi}_y & \bar{\xi}_\xi & \bar{\xi}_\eta \\ \bar{\eta}_x & \bar{\eta}_y & \bar{\eta}_\xi & \bar{\eta}_\eta \end{pmatrix}$$

$$=X_2X_1.$$

Clearly this can be extended to lenses consisting of any number of components.

3. Transformation of the X-Matrix

So far we have been dealing with the variables x, y, ξ, and ζ. It is convenient to apply a transformation, **T**, mapping these variables into a new system of coordinates, say α, β, γ, and δ, given formally by writing the latter as continuously differentiable functions of the former. The same transformation is applied to both the object and image space variables.

If we designate the M-matrix by

$$M=\begin{pmatrix} \alpha'_\alpha & \alpha'_\beta & \alpha'_\gamma & \alpha'_\delta \\ \beta'_\alpha & \beta'_\beta & \beta'_\gamma & \beta'_\delta \\ \gamma'_\alpha & \gamma'_\beta & \gamma'_\gamma & \gamma'_\delta \\ \delta'_\alpha & \delta'_\beta & \delta'_\gamma & \delta'_\delta \end{pmatrix},$$

then

$$M=A'XA^{-1},$$

where

$$A=\begin{pmatrix} \alpha_x & \alpha_y & \alpha_\xi & \alpha_\eta \\ \beta_x & \beta_y & \beta_\xi & \beta_\eta \\ \gamma_x & \gamma_y & \gamma_\xi & \gamma_\eta \\ \delta_x & \delta_y & \delta_\xi & \delta_\eta \end{pmatrix}.$$

A' represents the same matrix but in image-space variables. Applying this transformation to equation (4) we have

$$(\bar{A}^{-1}\bar{X}\bar{A}')\bar{A}'^{-1}JA'^{-1}(A'XA^{-1})=\bar{A}^{-1}JA^{-1},$$

which reduces to

$$\bar{M}(\bar{A}'^{-1}JA'^{-1})M=(\bar{A}'^{-1}JA^{-1}). \tag{8}$$

If the transformation, **T**, is restricted by the equations

$$\bar{A}JA=J, \quad \bar{A}'JA'=J, \tag{9}$$

then (8) becomes

$$\bar{M}\,J\,M=J, \tag{10}$$

which is identical in form to (4) except that the variables are changed from x, y, ξ, and η to α, β, γ, and δ. Thus we can change our variables in any way that is convenient, provided that (9) is satisfied, without adding to the complexity of the fundamental optical invariant.

4. Rotational Symmetric Systems

In the case of rotationally symmetric systems we are in a position to make a particularly advantageous choice of one of the new variables. In such a system the skewness invariant holds and

$$p=x'\eta'-y'\xi'=x\eta-y\xi.$$

By letting $\gamma=x\eta-y\xi=p$, and $\gamma'=x'\eta'-y'\xi'$, the M-matrix degenerates to

$$M=\begin{pmatrix} \alpha'_\alpha & \alpha'_\beta & \alpha'_p & \alpha'_\delta \\ \beta'_\alpha & \beta'_\beta & \beta'_p & \beta'_\delta \\ 0 & 0 & 1 & 0 \\ \delta'_\alpha & \delta'_\beta & \delta'_p & \delta'_\delta \end{pmatrix}.$$

When this is substituted into (10) we obtain the transformed fundamental optical invariant

$$\left.\begin{aligned} \alpha'_\alpha &= 1+\beta'_p\delta'_\alpha-\beta'_\alpha\delta'_p \\ \alpha'_\beta &= \beta'_p\delta'_\beta-\beta'_\beta\delta'_p \\ \alpha'_\delta &= \beta'_p\delta'_\delta-\beta'_\delta\delta'_p \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} \beta'_\alpha\delta'_\beta-\beta'_\beta\delta'_\alpha &= 0 \\ \beta'_\alpha\delta'_\delta-\beta'_\delta\delta'_\alpha &= 0 \\ \beta'_\beta\delta'_\delta-\beta'_\delta\delta'_\beta &= 1. \end{aligned}\right\} \tag{12}$$

We may consider the first two equations of (12) as a linear, homogeneous, simultaneous algebraic system in β'_α and δ'_α. A necessary and sufficient condition for a non-trivial solution to exist is that the determinant of coefficients must vanish identically. However, the third equation of (12) shows that this determinant is identical to a non-zero constant. We conclude that $\beta'_\alpha=\delta'_\alpha=0$. Therefore the first equation of (11) reduces to $\alpha'_\alpha=1$. The remaining equations of the transformed fundamental optical invariant become

$$\left.\begin{aligned} \alpha'_\beta &= \beta'_p\delta'_\beta-\beta'_\beta\delta'_p \\ \alpha'_\delta &= \beta'_p\delta'_\delta-\beta'_\delta\delta'_p, \end{aligned}\right\} \tag{13}$$

and

$$\beta'_\beta\delta'_\delta-\beta'_\delta\delta'_\beta=1. \tag{14}$$

Solving (13) for β'_p and δ'_p, using (14), we obtain equivalent expressions

$$\left.\begin{aligned}\beta'_p&=\beta'_\beta\alpha'_\delta-\beta'_\delta\alpha'_\beta\\ \delta'_p&=\delta'_\beta\alpha'_\delta-\delta'_\delta\alpha'_\beta.\end{aligned}\right\} \tag{15}$$

Thus, under the assumption of rotational symmetry the M-matrix reduces to

$$M=\begin{pmatrix}1 & \alpha'_\beta & \alpha'_p & \alpha'_\delta\\ 0 & \beta'_\beta & \beta'_p & \beta'_\delta\\ 0 & 0 & 1 & 0\\ 0 & \delta'_\beta & \delta'_p & \delta'_\delta\end{pmatrix}.$$

By applying (15) this can be factored into

$$M=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \beta'_\beta & 0 & \beta'_\delta\\ 0 & 0 & 1 & 0\\ 0 & \delta'_\beta & 0 & \delta'_\delta\end{pmatrix}\begin{pmatrix}1 & \alpha'_\beta & \alpha'_p & \alpha'_\delta\\ 0 & 1 & \alpha'_\delta & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -\alpha'_\beta & 1\end{pmatrix}. \tag{16}$$

5. Further Simplification

Suppose we have a rotationally-symmetric optical system defined by the functions α', β' and δ' of α, β, p and δ, consisting of two components, the first defined by α', β', and δ' of $\bar{\alpha}$, $\bar{\beta}$, $\bar{p}$, and $\bar{\delta}$ and the second by $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\delta}$ of α, β, p and δ. Of course p and $\bar{p}$ are identical numerically but play different logical roles which must be kept distinct. As indicated in (6),

$$\begin{pmatrix}1 & \alpha'_\beta & \alpha'_p & \alpha'_\delta\\ 0 & \beta'_\beta & \beta'_p & \beta'_\delta\\ 0 & 0 & 1 & 0\\ 0 & \delta'_\beta & \delta'_p & \delta'_\delta\end{pmatrix}=\begin{pmatrix}1 & \alpha'_{\bar{\beta}} & \alpha'_{\bar{p}} & \alpha'_{\bar{\delta}}\\ 0 & \beta'_{\bar{\beta}} & \beta'_{\bar{p}} & \beta'_{\bar{\delta}}\\ 0 & 0 & 1 & 0\\ 0 & \delta'_{\bar{\beta}} & \delta'_{\bar{p}} & \delta'_{\bar{\delta}}\end{pmatrix}\begin{pmatrix}1 & \bar{\alpha}_\beta & \bar{\alpha}_p & \bar{\alpha}_\delta\\ 0 & \bar{\beta}_\beta & \bar{\beta}_p & \bar{\beta}_\delta\\ 0 & 0 & 1 & 0\\ 0 & \bar{\delta}_\beta & \bar{\delta}_p & \bar{\delta}_\delta\end{pmatrix}. \tag{17}$$

By interchanging the two right columns and the two bottom rows of each matrix in (17) we obtain

$$\begin{pmatrix}1 & \alpha'_\beta & \alpha'_\delta & \alpha'_p\\ 0 & \beta'_\beta & \beta'_\delta & \beta'_p\\ 0 & \delta'_\beta & \delta'_\delta & \delta'_p\\ 0 & 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \alpha'_{\bar{\beta}} & \alpha'_{\bar{\delta}} & \alpha'_{\bar{p}}\\ 0 & \beta'_{\bar{\beta}} & \beta'_{\bar{\delta}} & \beta'_{\bar{p}}\\ 0 & \delta'_{\bar{\beta}} & \delta'_{\bar{\delta}} & \delta'_{\bar{p}}\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & \bar{\alpha}_\beta & \bar{\alpha}_\delta & \bar{\alpha}_p\\ 0 & \bar{\beta}_\beta & \bar{\beta}_\delta & \bar{\beta}_p\\ 0 & \bar{\delta}_\beta & \bar{\delta}_\delta & \bar{\delta}_p\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{18}$$

Now let

$$A'=(\alpha'_\beta,\alpha'_\delta),\qquad B'=(\alpha'_p),$$

$$C'=\begin{pmatrix}\beta'_\beta\beta'_\delta\\ \delta'_\beta\delta'_\delta\end{pmatrix},\qquad D'=\begin{pmatrix}\beta'_p\\ \delta'_p\end{pmatrix},$$

with $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ and A, B, C, D defined similarly. Then, by partitioning the matrices of (18) we obtain

$$\begin{pmatrix} 1 & A' & B' \\ e & C' & D' \\ 0 & f & 1 \end{pmatrix} = \begin{pmatrix} 1 & \overline{A} & \overline{B} \\ e & \overline{C} & \overline{D} \\ 0 & f & 1 \end{pmatrix} \begin{pmatrix} 1 & A & B \\ e & C & D \\ 0 & f & 1 \end{pmatrix} \tag{19}$$

where $e=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $f=(0,0)$. Then from (19) we can obtain

$$C'=\overline{C}C$$

or

$$\begin{pmatrix} \beta'_\beta & \beta'_\delta \\ \delta'_\beta & \delta'_\delta \end{pmatrix} = \begin{pmatrix} \beta'_{\bar{\beta}} & \beta'_{\bar{\delta}} \\ \delta'_{\bar{\beta}} & \delta'_{\bar{\delta}} \end{pmatrix} \begin{pmatrix} \bar{\beta}_\beta & \bar{\beta}_\delta \\ \bar{\delta}_\beta & \bar{\delta}_\delta \end{pmatrix}. \tag{20}$$

It has been shown that a consequence of (11) is that $\alpha'_a=1$, whence

$$\alpha'=\alpha+\lambda'(\beta, p, \delta). \tag{21}$$

Applying this relation to the two components of the system under consideration,

$$\alpha'=\bar{\alpha}+\bar{\lambda}(\bar{\beta}, \bar{p}, \bar{\delta})$$

$$\bar{\alpha} =\alpha+\lambda(\beta, p, \delta).$$

Thus

$$\alpha'-\alpha=\lambda'=\bar{\lambda}+\lambda. \tag{22}$$

Thus the geometrical properties of a rotationally-symmetric optical system can be described completely by the system of equations

$$\alpha'-\alpha=\sum_{i=1}^{n} \lambda_i \tag{23}$$

and

$$\begin{pmatrix} \beta'_\beta & \beta'_\delta \\ \delta'_\beta & \delta'_\delta \end{pmatrix} = \prod_{i=1}^{n} \begin{pmatrix} \partial\beta_i/\partial\beta_{i-1} & \partial\beta_i/\partial\delta_{i-1} \\ \partial\delta_i/\partial\beta_{i-1} & \partial\delta_i/\partial\delta_{i-1} \end{pmatrix}, \tag{24}$$

where λ_{i+1} is a function of β_i, p and δ_i, where $\beta'=\beta_n$, $\delta'=\delta_n$, and where $\beta=\beta_0$, $\delta=\delta_0$. This is a system of five scalar equations of which four are independent by (14).

If we replace the left members of these equations by functions representing a perfect optical system we are left with equations of condition on the parameter occurring in each of the components of the system. Alternatively a measure of the failure of the equations to be satisfied can be taken as a measure of the aberrations of the optical system.

6. Functions for a Perfect Optical System.

A possible choice for the variables, suitable for work with an optical system with object and image planes located a finite distance from the lens is

$$\left.\begin{aligned} \alpha&=\arctan y/x, \\ \beta&=\sqrt{x^2+y^2}, \\ \delta&=\frac{x\xi+y\eta}{\sqrt{x^2+y^2}}. \end{aligned}\right\} \tag{25}$$

It can be verified that these satisfy (9) when $\gamma=p$.

By integrating the equations for a perfect optical system (1) and (2), we obtain $x'=mx$, $y'=my$, the constant of integration being eliminated by a suitable choice of coordinate axes.

From this

$$\alpha'=\alpha, \text{ and } \beta'=m\beta,$$

whence $\beta'_\beta=m$ and $\beta'_\delta=0$. Substituting this in (14) we obtain

$$m\delta'_\delta=1.$$

Integrating:

$$\delta'=\delta/m+\psi,$$

where ψ is an arbitrary function of β. Thus for a perfect optical system described by the variables in (25) we have

$$\alpha'-\alpha=\sum_i \lambda_i=0, \tag{26}$$

and

$$\begin{pmatrix} m & 0 \\ \psi_\beta & 1/m \end{pmatrix}=\prod_i \begin{pmatrix} \partial\beta_i/\partial\beta_{i-1} & \partial\beta_i/\partial\delta_{i-1} \\ \partial\delta_i/\partial\beta_{i-1} & \partial\delta_i/\partial\delta_{i-1} \end{pmatrix} \tag{27}$$

as our equations of condition. It can be shown that ψ_β when $\beta=0$ equals the reciprocal of the focal length of the perfect lens.

The complexity of the problem may be reduced somewhat by the following considerations. Suppose (27) were satisfied for meridian rays only. Then

$$\begin{aligned} \beta' &= m\beta+\epsilon_1(\beta,\delta,p) \\ \delta' &= \delta/m+\epsilon_2(\beta,\delta,p) \end{aligned}$$

where $\epsilon_1=\epsilon_2=0$ when $p=0$. If (26) is subsequently corrected for skew rays then $\alpha'_\beta=\alpha'_\delta=0$, which by (15) implies that $\beta'_p=\delta'_p=0$. Thus the derivatives of ϵ_1 and ϵ_2 with respect to p are zero. Therefore these two functions are independent of p. Since ϵ_1 and ϵ_2 vanish for a particular value of p, they must equal zero for all values and (27) is satisfied for skew rays as well as meridian rays. Thus, the solution of (27) for meridian rays and the solution of (26) for skew rays assures the correction of the optical system for skew rays.

7. Transfer and Refraction Calculations

The ultimate components of an optical system will be considered here to consist of (a) a transfer across a space between two planes and (b) a transfer from a plane (which will be called the reference plane) to a refracting surface, refraction across the surface and a transfer back to the reference plane. The ray tracing formulas for the transfer are

$$\begin{aligned} x' &= x+t\xi/\zeta \\ y' &= y+t\eta/\zeta \\ \xi' &= \xi \\ \eta' &= \eta \end{aligned}$$

where t is the distance between the two planes, whence

$$\left.\begin{aligned} \alpha' &= \alpha+\arctan\frac{tp}{\beta^2\zeta+t\beta\delta}, \\ &= \alpha+\arctan p/\beta\delta-\arctan p/\beta'\delta', \\ \beta' &= \sqrt{(\beta+t\delta/\zeta)^2+(tp/\beta\zeta)^2}, \\ \delta' &= \frac{\delta(\beta+t\delta/\zeta)+(tp/\beta\zeta)p/\beta}{\sqrt{(\beta+t\delta/\zeta)^2+(tp/\beta\zeta)^2}}. \end{aligned}\right\} \tag{28}$$

For meridian rays the latter two formulas are

$$\left.\begin{aligned}\beta' &= (\beta + t\delta/\zeta)\\ \delta' &= \delta,\end{aligned}\right\} \tag{29}$$

where $\zeta^2 = N^2 - \delta^2$. Accordingly the matrix representing transfer between two planes separated by a distance t, in a medium with index of refraction N, is

$$m_T = \begin{pmatrix} 1 & N^2 t/\zeta^3 \\ 0 & 1 \end{pmatrix}. \tag{30}$$

The ray tracing formulas for a transfer from a plane to a refracting surface, refraction across the surface, and a transfer from the surface back to the plane are, respectively

$$\left.\begin{aligned}\bar{x} &= x + \bar{z}\xi/\zeta\\ \bar{y} &= y + \bar{z}\eta\zeta,\\ \xi' &= \xi + \bar{x}\phi\\ \eta' &= \eta + \bar{y}\phi\\ \zeta' &= \zeta - \bar{\beta}\phi/\bar{z}_{\bar{\beta}}\\ x' &= \bar{x} - \bar{z}\xi'/\zeta'\\ y' &= \bar{y} - \bar{z}\xi'/\zeta',\end{aligned}\right\} \tag{31}$$

where $\bar{z}$ is a function of $\bar{\beta}$ denoting the refracting surface and where

$$\phi = \frac{\bar{z}_{\bar{\beta}}}{\bar{\beta}(1+\bar{z}_{\bar{\beta}}^2)}\left\{(\zeta - \bar{\delta}\bar{z}_{\bar{\beta}}) - \sqrt{(\zeta - \bar{\delta}\bar{z}_{\bar{\beta}})^2 - (1+\bar{z}_{\bar{\beta}}^2)(N^2 - N'^2)}\right\}. \tag{32}$$

In terms of the α, β, δ coordinate system, these are

$$\left.\begin{aligned}\bar{\alpha} &= \alpha + \arctan\frac{\bar{z}p}{\beta^2\zeta + \bar{z}\beta\delta},\\ \bar{\beta} &= \sqrt{(\beta + \bar{z}\delta/\zeta)^2 + (\bar{z}p/\beta\zeta)^2},\\ \bar{\delta} &= \frac{\delta(\beta + \bar{z}\delta/\zeta) + (p/\beta)(\bar{z}p/\beta\zeta)}{\sqrt{(\beta + \bar{z}\delta/\zeta)^2 + (\bar{z}p/\beta\zeta)^2}},\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned}\bar{\delta}' &= \bar{\delta} + \bar{\beta}\phi\\ \zeta' &= \zeta - \bar{\beta}\phi/\bar{z}_{\bar{\beta}}\end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned}\alpha' &= \bar{\alpha} - \arctan\frac{\bar{z}p}{\bar{\beta}^2\zeta' - \bar{z}\bar{\beta}\bar{\delta}'} = \bar{\alpha} - \arctan\frac{p}{\beta'\delta'} + \arctan\frac{p}{\beta\delta'},\\ \beta' &= \sqrt{(\bar{\beta} - \bar{z}\bar{\delta}'/\zeta')^2 + (\bar{z}p/\bar{\beta}\zeta')^2}\\ \delta' &= \frac{\bar{\delta}'(\bar{\beta} - \bar{z}\bar{\delta}'/\zeta') - (p/\bar{\beta})(\bar{z}p/\bar{\beta}\zeta')}{\sqrt{(\bar{\beta} - \bar{z}\bar{\delta}'/\zeta')^2 + (\bar{z}p/\bar{\beta}\zeta')^2}}.\end{aligned}\right\} \tag{35}$$

Thus for skew rays

$$\alpha' = \alpha - \arctan \frac{p}{\beta' \delta'} + \arctan \frac{p}{\bar{\beta}\bar{\delta}'} - \arctan \frac{p}{\bar{\beta}\bar{\delta}} + \arctan \frac{p}{\beta\delta}$$

$$= \alpha - \arctan \frac{p}{\beta' \delta'} + \arctan \frac{p\bar{\beta}(\bar{\delta} - \bar{\delta}')}{\bar{\beta}^2 \bar{\delta}\bar{\delta}' + p^2} + \arctan \frac{p}{\beta\delta}$$

$$= \alpha - \arctan \frac{p}{\beta' \delta'} - \arctan \frac{p\phi}{\bar{\beta}\bar{\delta}\phi + (\bar{\delta}^2 + p^2/\bar{\beta}^2)} + \arctan \frac{p}{\beta\delta}. \tag{36}$$

For meridian rays the formulas for β and δ reduce from those given in (33), (34), and (35) to

$$\left.\begin{aligned} \bar{\beta} &= \beta + \bar{z}\delta/\zeta \\ \bar{\delta} &= \delta \end{aligned}\right\}, \tag{37}$$

$$\left.\begin{aligned} \bar{\beta}' &= \bar{\beta} \\ \bar{\delta}' &= \bar{\delta} + \bar{\beta}\phi, \end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned} \beta' &= \bar{\beta}' - \bar{z}\bar{\delta}'/\zeta' \\ \delta' &= \bar{\delta}' \end{aligned}\right\}, \tag{39}$$

where $\zeta^2 = N^2 - \delta^2$ and $\zeta'^2 = N'^2 - \delta'^2$.

By differentiating the first equation of (37) with respect to β and δ, recalling that $\bar{z}$ is a function of $\bar{\beta}$, we obtain

$$\bar{\beta}_\beta = 1 + \bar{z}_{\bar{\beta}}\bar{\beta}_\beta\delta/\zeta$$
$$\bar{\beta}_\delta = \bar{z}/\zeta + \bar{z}_{\bar{\beta}}\bar{\beta}_\delta\delta/\zeta + \bar{z}\delta^2/\zeta^3$$

so that

$$\bar{\beta}_\beta = \zeta/\Delta$$
$$\bar{\beta}_\delta = (\zeta/\Delta)\,(N^2\bar{z}/\zeta^3),$$

where

$$\Delta = \zeta - \bar{z}_{\bar{\beta}}\delta.$$

Thus

$$m_a = \begin{pmatrix} \bar{\beta}_\beta & \bar{\beta}_\delta \\ \bar{\delta}_\beta & \bar{\delta}_\delta \end{pmatrix} = \begin{pmatrix} \zeta/\Delta & (\zeta/\Delta)\,(N^2\bar{z}/\zeta^3) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \zeta/\Delta & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & N^2\bar{z}/\zeta^3 \\ 0 & 1 \end{pmatrix}$$

By differentiating the equations of (38),

$$m_b = \begin{pmatrix} \bar{\beta}'_{\bar{\beta}} & \bar{\beta}'_{\bar{\delta}} \\ \bar{\delta}'_{\bar{\beta}} & \bar{\delta}'_{\bar{\delta}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \bar{\beta}\zeta'\bar{z}_{\bar{\beta}\bar{\beta}}\phi/\bar{z}_{\bar{\beta}}\Delta' & 1 + \bar{\beta}\phi(\bar{\delta} + \bar{z}_{\bar{\beta}}\zeta)/\Delta'\zeta \end{pmatrix},$$

where $\Delta' = \zeta' - \bar{z}_{\bar{\beta}}\bar{\delta}'$. From the definitions of Δ and Δ' and from (34) it can be shown that $1 + \bar{\beta}\phi(\bar{\delta} + \bar{z}_{\bar{\beta}}\zeta)/\Delta'\zeta = \Delta\zeta'/\Delta'\zeta$, resulting in a simplification of m_b.

Again, differentiating (39) we obtain

$$m_c=\begin{pmatrix}\beta'\bar{\beta}' & \beta'\bar{\delta}'\\ \delta'\bar{\beta}' & \delta'\bar{\delta}'\end{pmatrix}=\begin{pmatrix}\Delta'/\zeta' & -\bar{\bar{z}}N'^2/\zeta'^3\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & -\bar{\bar{z}}N'^2/\zeta'^3\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Delta'/\zeta' & 0\\ 0 & 1\end{pmatrix}.$$

When these three matrices are multiplied together, we have

$$m_R=m_c m_b m_a$$

$$=\begin{pmatrix}1 & -\bar{\bar{z}}N'^2/\zeta'^3\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Delta'\zeta/\Delta\zeta' & 0\\ \bar{\beta}\zeta\zeta'\bar{\bar{z}}_{\bar{\beta}\bar{\beta}}\phi/\bar{\bar{z}}_{\bar{\beta}}\Delta\Delta' & \Delta\zeta'/\Delta'\zeta\end{pmatrix}\begin{pmatrix}1 & \bar{z}N^2/\zeta^3\\ 0 & 1\end{pmatrix}, \tag{40}$$

Note that the first and last factors are formally identical to m_T and can therefore be combined conveniently with an adjoining transfer matrix.

When the refracting surface is a sphere centered on the reference plane

$$\bar{\bar{z}}=-\frac{1}{c}\sqrt{1-c^2\bar{\beta}^2}.$$

The minus sign assures that the sphere is concave to the right when c is positive, in accordance with the usual convention. Then

$$m_R=\begin{pmatrix}1 & -\bar{\bar{z}}N''^2/\zeta'^3\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Delta'\zeta/\Delta\zeta' & 0\\ \zeta\zeta'\phi/c^2\bar{\bar{z}}^2\Delta\Delta' & \Delta\zeta'/\Delta'\zeta\end{pmatrix}\begin{pmatrix}1 & \bar{\bar{z}}N^2/\zeta^3\\ 0 & 1\end{pmatrix}, \tag{41}$$

where

$$\left.\begin{aligned}\Delta&=(\bar{\bar{\beta}}\bar{\bar{\delta}}+\bar{\bar{z}}\zeta)/\bar{\bar{z}},\\ \Delta'&=(\bar{\bar{\beta}}\bar{\bar{\delta}}'+\bar{\bar{z}}\zeta')/\bar{\bar{z}}=\sqrt{\Delta^2-\frac{1}{c^2\bar{\bar{z}}}(N^2-N'^2)},\\ \phi&=-c^2\bar{\bar{z}}(\Delta-\Delta').\end{aligned}\right\} \tag{42}$$

8. A Simple Illustration

As an example we consider the properties of a single refracting spherical surface. Consider an optical system consisting of an object plane, a spherical surface separating media of refractive indices N and N' and an image plane. Let the object and image planes be located distances t and t' from the center of the spherical surface. Figure 1 shows the case where t,

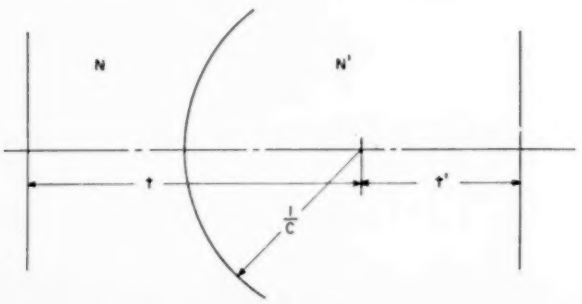

FIGURE 1

t', and c are positive. From (27), (30), and (41) the equation of condition for meridian rays is

$$\begin{pmatrix} m & 0 \\ \psi_\beta & 1/m \end{pmatrix}$$

$$= \begin{pmatrix} 1 & N'^2t'/\zeta'^3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\bar{z}N'^2/\zeta'^3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta'\zeta/\Delta\zeta' & 0 \\ \zeta\zeta'\phi/c^2\bar{z}^2\Delta\Delta' & \Delta\zeta'/\Delta'\zeta \end{pmatrix} \begin{pmatrix} 1 & \bar{z}N^2/\zeta^3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & N^2t/\zeta^3 \\ 0 & 1 \end{pmatrix}. \tag{43}$$

For skew rays, by (26), (28), and (36),

$$\alpha'-\alpha = -\arctan\frac{p}{\beta'\delta'} + \arctan\frac{p}{\beta\delta} - \arctan\frac{p\phi}{\beta\delta\phi + (\delta^2 + p^2/\beta^2)} = 0. \tag{44}$$

Expressing (43) in scalar form we obtain four equations.

$$\left.\begin{aligned} \Delta'\zeta/\Delta\zeta' + N'^2(t'-\bar{z})\zeta\phi/c^2\bar{z}^2\Delta\Delta'\zeta'^2 &= m \\ \zeta\zeta'\phi/c^2\bar{z}^2\Delta\Delta' &= \psi_\beta \\ \frac{N^2(t+\bar{z})\Delta'}{\Delta\zeta^2\zeta'} + \frac{N'^2(t'-\bar{z})\Delta}{\Delta'\zeta\zeta'^2} + \frac{N^2(t+\bar{z})N'^2(t'-\bar{z})\phi}{c^2\bar{z}^2\Delta\Delta'\zeta^2\zeta'^2} &= 0 \\ \Delta\zeta'/\Delta'\zeta + N^2(t+\bar{z})\zeta'\phi/c^2\bar{z}^2\Delta\Delta'\zeta^2 &= 1/m. \end{aligned}\right\} \tag{45}$$

We will pay particular attention to the third of these, which we write in the form

$$N^2(t+\bar{z})\Delta'^2\zeta' + N'^2(t'-\bar{z})\Delta^2\zeta + (1/c^2\bar{z}^2)(t+\bar{z})N'^2(t'-z)\phi = 0. \tag{46}$$

Let

$$\sigma = \Delta\bar{z} = \beta\delta + \bar{z}\zeta,$$

$$\sigma' = \Delta'\bar{z} = \beta\delta' + \bar{z}\zeta' = \sqrt{\sigma^2 - (1/c^2)(N^2 - N'^2)}.$$

Also,

$$\zeta' = \zeta + \bar{z}\phi,$$

where $\phi = -c^2(\sigma-\sigma')$ by (42). Substituting these yields

$$c^2\sigma^2\zeta[N'^2(t'-\bar{z}) + N^2(t+\bar{z})] - N^2\zeta(t+\bar{z})(N^2-N'^2) + N^2(t+\bar{z})[c^2\sigma^2\bar{z} - N^2\bar{z} + N'^2t']\phi = 0.$$

This may be written as

$$c^2\sigma^2\zeta(N'^2t' + N^2t) + \zeta(N^2-N'^2)[c^2\sigma^2\bar{z} - N^2(t+\bar{z})] + N^2(t+\bar{z})[c^2\sigma^2\bar{z} - N^2\bar{z} + N'^2t']\phi = 0.$$

If

$$N'^2t' + N^2t = 0, \tag{47}$$

Then

$$[\zeta(N^2-N'^2) + N^2(t+\bar{z})\phi][c^2\sigma^2\bar{z} - N^2(t+\bar{z})] = 0.$$

We thus obtain two possible cases represented by the two equations

$$\zeta(N^2-N'^2) + N^2(t+\bar{z})\phi = 0 \tag{48}$$

$$c^2\sigma^2\bar{z} - N^2(t+\bar{z}) = 0. \tag{49}$$

When $\bar{\beta}=\bar{\delta}=0$,

$$\bar{z}=-1/c, \sigma=-N/c, \sigma'=-N'/c, \zeta=N, \text{ and } \phi=c(N-N').$$

Substituting in (48) we obtain

$$N(N^2-N'^2)+N^2(t-1/c)c(N-N')=0,$$

$$N+N'+N(ct-1)=0,$$

$$t=-N'/Nc. \tag{50}$$

Applying this to (47),

$$t'=N/N'c. \tag{51}$$

When $\bar{\beta}$ and $\bar{\delta}$ are not zero, it can be shown that (50) and (51) satisfy (48) for all values of δ provided that $\beta=0$. Therefore only the axial point on the object plane is imaged perfectly. This result provides the aplanatic points of a sphere [9] (see fig. 2).

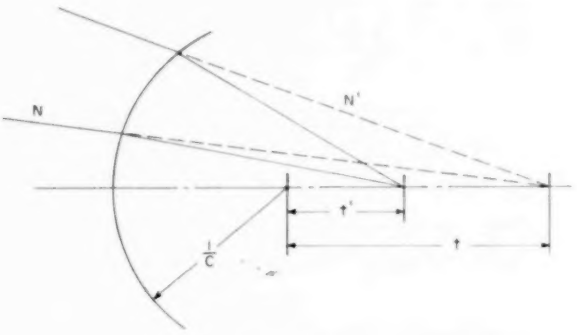

FIGURE 2

On making the same substitutions into (49) we find that $t=0$, which when applied to (47) yields $t'=0$. As in the previous case, these are true for all values of $\bar{\beta}$ and $\bar{\delta}$ provided that $\beta=0$. This illustrates the well-known fact that the center of a sphere is imaged perfectly on itself [4].

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Analytical Integration of the Differential Equation for Water Storage*

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The integration of the storage differential equation at present is usually done mostly by graphical or numerical procedures. An approach to the analytical integration of that equation is the subject of this paper. A new method of fitting the given background curves by mathematically tractable expressions is introduced. The storage-outflow discharge relation is expressed in the form of a power function. A general differential equation for water storage $y'+cP_t y^2-cy^k=0$ is derived, with c and k constants for the given reservoir, outflow shape and type of flow, and P being the inflow hydrograph. The analytical solutions of this equation for $P=0$, $P=$ constant, and certain $P=f(t)$ are given for the integrable cases (tables 1 to 3, eqs (12) to (29)). The application of the results obtained is discussed at the end of the paper.

1. Introduction

The known simple balance equation, with the inflow minus the outflow in a time interval equal to the storage change in that interval is

$$P-Q=\frac{dS}{dt} \tag{1}$$

which is normally derived from the definition itself; with P the inflow, Q the outflow, S the storage volume, and t the time.

The two basic differential equations which govern the unsteady water movement through lakes and channels are the continuity and the dynamic (or momentum) equations:

$$\frac{\partial(Av)}{\partial x}+\frac{\partial A}{\partial t}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial h}{\partial x}=gS_f \tag{3}$$

where

A = cross-sectional flow area,
v = mean velocity of flow,
x = length along the lake or the channel,
t = time,
g = earth acceleration,
h = level of water surface in a cross section with reference to a fixed level, and
S_f = frictional slope.

The applications of eqs (2) and (3) must satisfy the basic assumptions made by De Saint Venant in their derivation, and specifically that the unsteady flow is gradually varied.

The eq (3) is omitted in many engineering problems, as in the routing of floods through great reservoirs and along channels of small slopes with unsteady flow very gradually varied, so that only eq (2) is used.

Two important problems which have not as yet been completely discussed, and are not considered here, are: (a) Under what conditions can eqs (2) and (3) be used, and (b) when can eq (3) be neglected without sensible error.

The first term of eq (2), $\partial(Av)/(\partial x)=(\partial Q)/(\partial x)$, can, for suitable small Δx, be replaced by $(\Delta Q)/(\Delta x)$; for a given time interval Δt and a reach of length Δx, can be replaced by $\Delta Q/\Delta x=(P-Q)/\Delta x$, where P is arrival or inflow discharge and Q is the departure or outflow discharge in m^3/s or in cfs. (Here P and Q are used instead of usual I and O.) The second term $(\partial A)/(\partial t)$ can be replaced for small Δt by $(\Delta A)/(\Delta t)=(\Delta A_1+\Delta A_2)/(2\Delta t)$, where $\Delta A_1=A_1''-A_1'$ and $\Delta A_2=A_2''-A_2'$, and ΔA_1 and ΔA_2 are changes of cross-sectional areas at the beginning, and at the end of the reach Δx during time Δt, and A_1' and A_2' are areas for the beginning, and A_1'' and A_2'' areas for the end of time interval Δt. Equation (2) can thus be expressed as

$$P-Q=\frac{\Delta A_1+\Delta A_2}{2}\frac{\Delta x}{\Delta t}=\frac{\Delta S}{\Delta t},$$

where ΔS is the volume change for the reach Δx during time interval Δt. If now $\Delta t\to 0$, this differ-

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ential equation becomes eq (1), in which S is the storage volume for body of water of the considered reach Δx, and dS/dt is the rate of change with time of this volume. In the case of a reservoir or lake, with small velocities along the lake and with inflow coming from different tributaries, the reach Δx is replaced by the water body of the lake. Strictly speaking, some end portions of lakes with shallow water and high inflow should be considered separately, but their influence on the outflow hydrograph can be neglected in comparison with the main influence of great fluctuating storage space. Equations (2) and (1) are identical, and the equation of continuity in its form of eq (1) will be called the *storage differential equation* for water in lakes and along channels of small slopes with unsteady flow very gradually varied with time.

The analytical integration of the storage differential equation would be useful in the solution of problems in cases where its terms can be approximated by integrable expressions. Some of these cases are: (a) Routing of very gradually varied floods along large channels of small bottom slope; (b) effect of unregulated lakes on flood waves; (c) routing of flood peaks through reservoirs for the inflow discharges higher than that discharge of hydrograph, which corresponds to the full capacity of open gates and valves; (d) study of the genesis of hydrograph and separation of water flow according to its origin (surface runoff, underground storage, lakes, channel storage, etc.); (e) outflow through partial openings in case of rupture of dam with no influence of tailrace levels on the outflow hydrograph, when the wave movement along the lake created by rupture can be neglected; (f) computation of the seepage water out of reservoirs, etc.

Equation (1) serves generally for the computation of relations between five functions:

(1) Inflow hydrograph, $P=f_1(t)$;
(2) outflow hydrograph, $Q=f_2(t)$;
(3) stage hydrograph, $H=f_3(t)$;
(4) outflow rating curve, $Q=f_4(H)$;
(5) storage function, $S=f_5(H)$; or area function $A=f_6(H)$, with five variables, Q, P, H, S, t.

When three of five functions with boundary conditions are known (three variables can be excluded), the eq (1) enables the computation of the relation between two remaining variables.

The usual integration procedures are numerical, graphical, combined numerical and graphical, and integration by special devices.

The analytical integration of eq (1) is not usually feasible in practice, mostly because of the difficulties of fitting easily three known curves out of five by tractable and integrable mathematical expressions.

The most common cases in the application of eq (1) have, as given, the following curves: (1) Storage or area function (obtained by survey); (2) outflow rating curve (obtained by gauging, hydraulic computation, model study, etc.); and (3) inflow (or outflow) hydrograph. Two other functions are normally to be computed; (1) outflow (or inflow) hydrograph, and (2) stage hydrograph.

The subject of this paper is the analytical integration of eq (1) for those cases for which inflow hydrograph, storage function and outflow rating curve can be entirely or partly fitted by simple expressions which make eq (1) integrable.

2. Fitting of Mathematical Expressions to Given Curves

2.1. Storage Function

The storage function, which relates lake or channel volume to its level referred to some datum, can be approximated either by the function (see references [1][1] to [5]) of the type

$$S=aH^m \tag{4}$$

or by the polynomial of the type

$$S=A_0+A_1H+A_2H^2+\ldots+A_mH^m \tag{4a}$$

where S=storage volume, H=depth of water above a reference level suitably selected, a and m, and A_0, A_1, A_2, . . ., A_m are coefficients to be determined from data of storage function.

The reference level of eq (4) is normally that of zero storage (lowest level of lake or reservoir, or river bed, etc.). The level of zero outflow does not coincide generally with the level of zero storage, and in that case another form of eq (4) is used:

$$S=a(H^m-H_0^m) \tag{4b}$$

where H_0=difference of the level of zero outflow and of the level of zero storage. The coefficients a and m in eq (4b) are the same as in eq (4). The use of eq (4), as will be shown later, gives a form to eq (1) which is integrable in many cases, but eq (4b) makes it less simple and more complex for analytical treatment.

Equation (4a) is well-suited to be used as the storage function, if enough terms $(m+1)$ are involved. It must be supposed that for the level $H=0$, the storage is A_0, so that eq (4a) can be used in case the level of zero outflow coincides with the level of storage A_0. If the level of zero outflow is changed, the coefficients A_0, A_1, A_2, . . ., A_m have also to be changed.

Instead of using eq (4b) repeatedly with various values of H_0 to cover the entire range of S versus H, a family of eqs (4) may be used with continuously changing a and m as function either of level of zero outflow or of H_c, the distance from level of zero outflow to a reference level. The coefficients a and m of eq (4) depend on level of zero outflow, or on difference of levels H_c, and are determined by the condition that $H=0$ of new zero storage coincides always with the level of zero outflow. This means that the coefficients a and m are to be determined for different levels of zero outflow (and therefore also of zero storage) as continuous functions. The

[1] Figures in brackets indicate the literature references at the end of this paper.

question arises whether eq (4) fits the thus conceived changing storage function as well as it fits the entire function from the lowest to the highest level of the body of water. In some cases eq (4), used in the manner described above, fits the upper parts of the storage curve better than the entire curve, but in other cases the reverse is true.

Figures 1 and 2 show the second case, and figures 3 and 4 the first case. Figures 1 and 3 show the storage functions $S=aH^m$ for the entire range of H=depth of water in reservoir at dam, and the functions $m=f_1(H_c)$ and $a=f_2(H_c)$ with H_c as distance of level of outflow zero to the sea level, or to the reservoir bottom at dam H'_c. Figures 2 and 4 show the fitting of eq (4) to the parts of storage curves for different values H_c. When H_c is expressed as H'_c, or as distance to the bottom of reservoir, it can be used as dimensionless number $h=H'_c/H_m$, with H_m=maximum depth of reservoir in the normal operation.

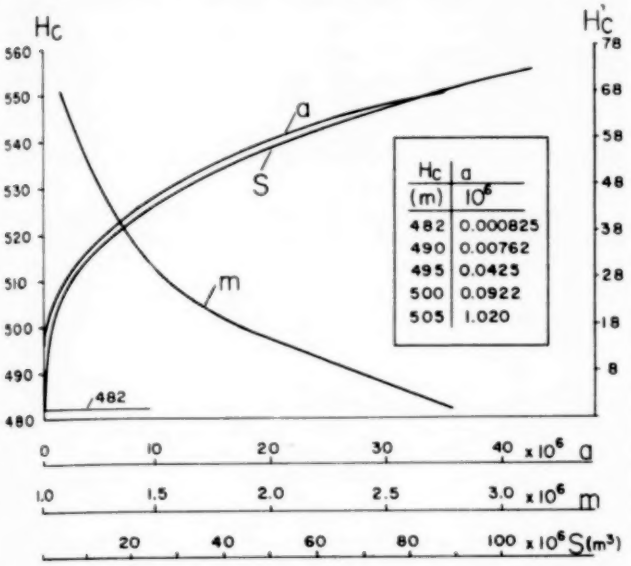

FIGURE 1. *Storage function $S=aH^m$, with a and m as functions of H_c, the levels of outflow zero (first example).*

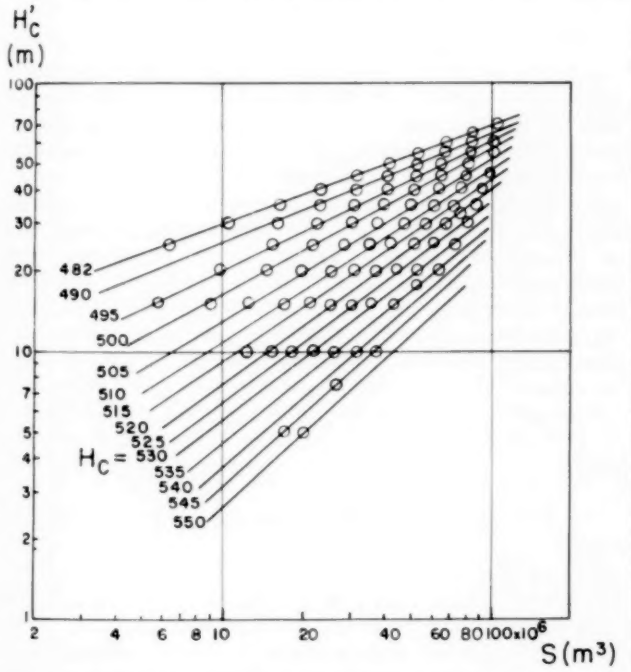

FIGURE 2. *Storage function of figure 1 in logarithmic scales for different values H_c of zero outflow, where the storage functions from H_c up to the highest level are fitted by the power functions $S=aH^m$ (first example).*

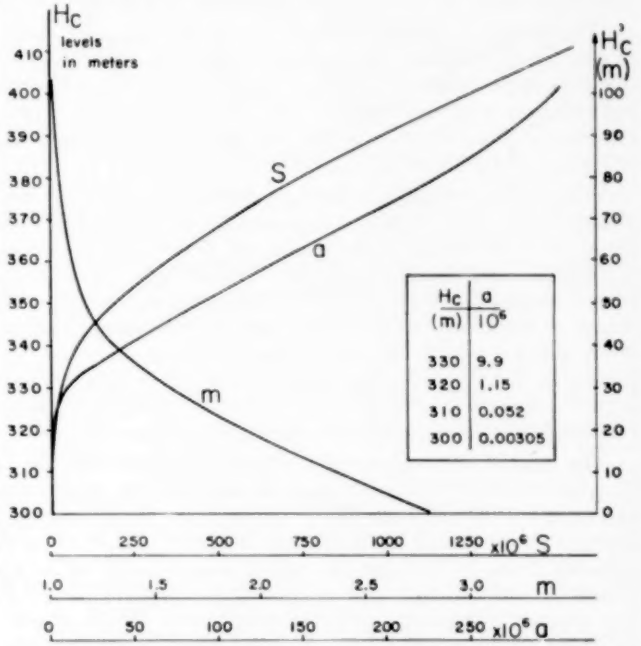

FIGURE 3. *Storage function $S=aH^m$, with a and m as functions of H_c, the levels of outflow zero (second example).*

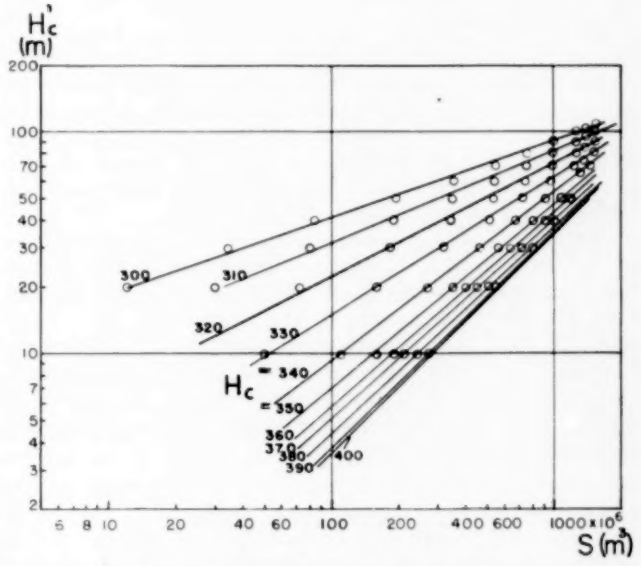

FIGURE 4. *Storage function of figure 3 in logarithmic scales for different values H_c of level of zero outflow, where the storage functions from H_c up to the highest level are fitted by the power functions $S=aH^m$ (second example).*

The storage functions in figures (1) and (3) are given from the reservoir bottom at dam up, and in figures 2 and 4 from the level of zero outflow up for each of given curves (as straight lines).

The function $m=f_1(H_e)$ decreases with higher value of H_e, i.e., higher level of zero outflow, and is approximately 1 for the highest levels and small range of level differences, where the storage function is nearly a straight line. For lower H_e and for higher range of storage levels, the value m is greater. The function $a=f_2(H_e)$ has the opposite characteristics.

If only one value of H_e is to be investigated, a and m are fixed coefficients, but if a range of H_e is to be investigated (dam-breaching problems with various breachings and different lowest outflow levels), the functions m and a of H_e can be easily obtained as in the case of figures 1 to 4.

When the sedimentation of reservoirs has to be taken into account, a and m are not only a function of level of zero outflow, but also of time (apart from sediment characteristics they depend on operation practice of reservoir, inflow hydrographs, etc.). As a and m make eq (4) less tractable in case they change with time (which is sometimes rather difficult to predict), they will be considered here as constants for a given reservoir, for given level of zero outflow, and for selected time period of reservoir life.

The coefficient m ranges within the limits 1 to 5, but for the majority of natural valleys the range is about 1 to 4. When H_e is high and the range of levels is very small, m is mostly 1.0 to 1.5 and rarely greater than 2; for highest range it is 2 to 5. The m depends not only on the range of levels, but also on the shape of reservoir cross sections. When those sections can be fitted by a power function for width of type $B=2pH^s$, m is function of s. The value a depends on H_e, but apart from that, a is higher for wider valleys, for lower river slopes, and for lower values of exponent s.

The property of the storage function to be well-fitted by an expression of the type of eq (4) can be used to compute this function in the case where only two points of the relation storage versus level are available. These two points can be found in the literature for practically any reservoir: capacities at highest and at lowest operational levels. Two pairs of values (S, H) are sufficient for the computation of values a and m in eq (4), thus allowing the determination of the curves of figures 1 and 3. Taking into account the errors in the survey of reservoirs and the fact that the storage function is being constantly changed by sedimentation and littoral erosion, the values a and m determined by eq (4) and the figures 1 to 4 are accurate enough for the analytical solution of practical problems.

2.2. Outflow Rating Curve

The outflow rating curves are relations of the departure discharge to the water level in the reservoir. They can be fitted also in many cases either by an expression of type

$$Q=bH^r, \tag{5}$$

or by a polynomial,

$$Q=b_0+b_1H+b_2H^2+b_3H^3+\ldots+b_rH^r, \tag{5a}$$

where Q=departure discharge, H=level above the depth for which $Q=b_0$, b and r or b_0, b_1, b_2, b_3, . . . b_r are coefficients depending only on boundary conditions (outlet shape) and on type of flow (closed or free outlets). The eq (5a) can be also conceived as a rating function expressed in power series form. The range of r in eq (5) for free surface outflow is 1.5—4.0 (usually 1.5 to 3.0) and for pressure type flow about $r=0.5$ in most cases. The range $r=0.5-1.5$ can be covered also in special cases for special shapes of outflows.

In case two types of flow, viz, flow under pressure and free surface flow, occur simultaneously, eq (5) takes the form

$$Q=b_1H_1^{r_1}+b_2H_2^{r_2}=b_1(H+H_a)^{r_1}+b_2H^{r_2}, \tag{5b}$$

where $H_2=H$ and $H_1=H+H_a$ are the hydraulic heads (level differences) to be taken for the former and latter types of flow respectively, b_1 and r_1 correspond to closed pressure flow and b_2 and r_2 to free surface flow. As r_1 and r_2 are not generally integers, eq (5b) is different from eq (5a).

The use of eqs (5a) and (5b) instead of eq (5) makes eq (1) integrable in a much smaller number of cases. Equation (5) will be used exclusively here.

2.3. Inflow Hydrograph

It is practically impossible to fit an entire natural inflow hydrograph by a single mathematical expression, but it is possible to fit some of their parts by tractable and simple functions.

The simplest forms of function $P=f(t)$ to be used for integration are: (a) $P=0$ (arrival discharge zero as dry period of year or as water fully stored in upstream reservoirs, etc.); (b) $P=P_0$=constant (or nearly constant, with low river flow slowly changing, regulated constant flow from upstream reservoirs, short-term operation of reservoirs, when flow could be placed by constant discharge); (c) $P=P_0-ft$ (gradually varied flow); $P=P_0t^{-s}$; $P=P_0e^{-ft}$; $P=P_0t^se^{-ft}$ (P_0, f, and s are different constants in each case).

2.4. Storage-Outflow Discharge Function

When the lowest outflow level coincides with the level of zero storage, or when reference levels for measuring H in both eqs (4) and (5) are the same (adjustment to this condition is always possible by taking the corresponding coefficients a and m from their functions, fig. 1 to 4), the elimination of H gives for the storage-outflow discharge function, or

relation $S=f(Q)$,

$$S=\frac{a}{b^{m/r}}\,Q^{m/r}=\frac{1}{c}\,Q^{n}, \tag{6}$$

where $c=b^{(m/r)}/a$ and $n=m/r$.

The coefficient n for free surface flow is in the range ¼ to 4, and for closed pressure flow 2 to 10, or a little greater.

The use of pairs of eqs (4) and (5a), (4) and (5b), (4a) and (5a), (4a) and (5b), (4b) and (5), (4b) and (5a), (4b) and (5b) gives a more complex relation $S=f(Q)$, in most cases in parametric form with H as the parameter, than the use of (4) and (5) or (4a) and (5). However, in this last case

$$S=a_0+\frac{1}{c_1}\,Q^{1/r}+\frac{1}{c_2}\,Q^{2/r}+\ldots+\frac{1}{c_m}\,Q^{m/r}, \tag{6a}$$

which makes eq (1) less tractable than the use of eqs (4) and (5).

As eqs (4) and (5) fit well many storage and outflow functions, for different value of H_c, and as eq (6) allows the integration of eq (1) in many cases, eqs (4) and (5), and (6) will be used in this study.

3. General Type of Differential Equations

With the introduction of eq (6) into eq (1), one obtains $n\,Q^{n-1}\,dQ+c(Q-P)\,dt=0$.

The substitution $y=Q^{-n}$ and the replacement $k=(2n-1)/n=2-\frac{1}{n}=2-\frac{r}{m}$ then gives the following expression:

$$y'+cPy^2-cy^k=0. \tag{7}$$

This is the general differential equation for storage reservoirs and storage channels with no artificial control of outlet flow, under conditions discussed previously in this paper. For free surface outflows the range of k is usually from about -2 for $n=¼$ to $k=¾$ for $n=4$, and for the closed type outflows the range of k is from ½ to 1 9/10 or a little greater, but less than the limit value $k=2$. Practically, the range of k is from -2 to $+2$, where $k=+2$ is not possible.

The relations of k or n to m and r are given in figure 5. The most common range for both m (1 to 4) and r (0.5 and 1.5 to 3) is specially emphasized for reservoirs in river valleys. In most practical cases $k>0$, and usually $k>1/2$.

The eq (7) could also be expressed as

$$y'+cy^2\,(P-1)-cy^{r/m}=0. \tag{8}$$

The general form of eqs (7) and (8) is

$$y'+My^h+Ny^g=0 \tag{9}$$

where M and N are functions of t; h and g are any real numbers. By the substitution $y=Z^{(1/(h-1))}$ the general form is reduced to

$$Z'+(h-1)MZ^2+(h-1)NZ^{\frac{h+g-2}{h-1}}=0 \tag{10}$$

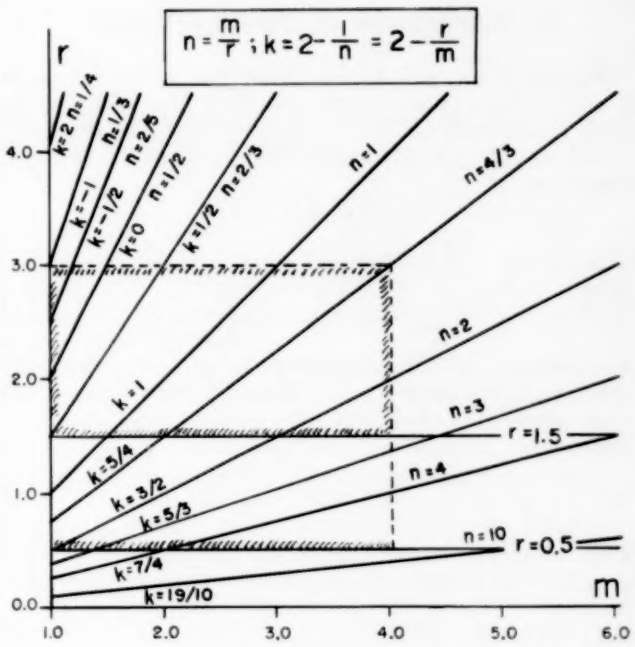

FIGURE 5. *Values n of storage-outflow discharge relationship* $S=\frac{1}{c}Q^n$, *and values* $k=2-\frac{1}{n}$, *in relation to m and r (exponents of storage and of rating curve, expressed in power form).*

which is eq (7) under conditions that $h-1=c$, $N=-1$, $M=P$, and $(h+g-2)/(h-1)=k$.

The general analytical solution of the ordinary differential equations of the type (7) or (10) for any k and for any expression of P does not exist. Chiellini [6] has shown that the solution is possible for $k=3$ if the expression $(M'N-MN')$ is different from M^3 by a constant factor. When k is a positive and integer number higher than 3, Luigi Conte [6, 7] has given the solution for special cases. As k is always less than two here, their procedures cannot be used for eq (7).

Both facts, (a) that k is a number derived from continuously changing natural conditions and is rarely an integer, and (b) that the function $P=f(t)$ is not simple and must be fitted by integrable expressions, make the analytical integration of eq (7) possible only in special cases. For $k=-2$, -1, 0, and 1 and some rational numbers the integration is simpler for some types of P, but the integration for intermediate values of k and for the majority of P functions has to be performed numerically or graphically. The analytical integration for special values of k and for selected functions $P=f(t)$ is useful, because the resulting functions give the type of curves which can be expected for intermediate values of k. Or, when k and outflow functions are known, some conclusions can be made about the inflow functions. The type of functions obtained in integrable cases can serve for control of curves obtained by other procedures.

4. Integration in Special Cases

Equation (7) will be analyzed for some values of k and some types of the function P. The values of k taken will be -2, -1, 0, 1 and, tentatively, $-3/2$, $-1/2$, $1/2$, $3/2$ along with some other rational numbers.

4.1. $P=0$

Equation (7) becomes

$$y'-cy^k=0, \tag{11}$$

with $y=y_0=Q_0^{-n}$ for $t=0$, where Q_0 is outflow discharge at the time $t=0$. The solution of eq (11) with $y=Q^{-n}$ is then

$$Q^{n-1}=Q_0^{n-1}-\left(1-\frac{1}{n}\right)ct. \tag{12}$$

This is the general expression for outflow hydrographs from storage reservoir with no inflow into the reservoir. The solution can be obtained from eq (12) for each value of n except for $n=1$ $(k=1)$, but eq (11) gives solution in this case directly (table 1). Therefore, for $P=0$, or when the constant inflow discharge is small and can be neglected in comparison to the outflow discharges, the analytical integration of eq (7) gives solutions for any $k(n)$, rational or irrational.

TABLE 1. $Q=Q_0\left(1-\frac{n-1}{n}\frac{ct}{Q_0^{n-1}}\right)^{\frac{1}{n-1}}$; $P=0$

k	n	Q
-2	$\frac{1}{4}$	$Q_0(1+3cQ_0^{\frac{3}{4}}t)^{-\frac{4}{3}}$
-1	$\frac{1}{2}$	$Q_0(1+2cQ_0^{\frac{3}{2}}t)^{-\frac{1}{2}}$
$-\frac{1}{2}$	$\frac{2}{5}$	$Q_0\left(1+\frac{3}{2}cQ_0^{\frac{3}{2}}t\right)^{-\frac{5}{3}}$
0	$\frac{1}{2}$	$Q_0(1+cQ_0^{\frac{1}{2}}t)^{-2}$
$\frac{1}{2}$	$\frac{2}{3}$	$Q_0\left(1+\frac{c}{2}Q_0^{\frac{1}{2}}t\right)^{-3}$
1	1	Q_0e^{-ct}
$\frac{4}{3}$	$\frac{3}{2}$	$Q_0\left(1-\frac{c}{3Q_0^{\frac{1}{2}}}t\right)^{2}$
$\frac{1}{2}$	2	$Q_0\left(1-\frac{c}{2Q_0}t\right)$
$\frac{5}{3}$	3	$Q_0\left(1-\frac{2c}{3Q_0^2}t\right)^{\frac{1}{2}}$
$\frac{7}{4}$	4	$Q_0\left(1-\frac{3c}{4Q_0^3}t\right)^{\frac{1}{3}}$

Equation (12) can be written in the form:

$$Q=Q_0\left(1-\frac{n-1}{n}\frac{ct}{Q_0^{n-1}}\right)^{\frac{1}{n-1}}. \tag{13}$$

This shows that the decrease of discharge ratio Q/Q_0 depends on both t and Q_0, except for $n=k=1$ (table 1), in which case it depends only on t.

The analytical expressions of eq (12) for some k (or n) values are given in table 1. Figure 6 gives curves for $n=\frac{1}{2}$, 1, $\frac{3}{2}$, 2 and 3, for given values of a and b, with $c=b^n/a$. The values a of eq (4) and b of eq (5) are taken constant for all five curves (for easier computation in examples), though they change when m and r (and so n) changes. For $n=2$ the outflow hydrograph is a straight line, which corresponds, among other cases, to the cylindric reservoir $(m=1)$ and to the outflow under pressure $(r=\frac{1}{2})$. The same type of linear hydrograph is obtained for $m=3$ and $r=\frac{3}{2}$, which is the case for some deep lakes with outflow function $Q=bH^{\frac{3}{2}}$. For river outlets from lakes, with $r\geq\frac{3}{2}$ and $m\leq 3$, $n\leq 2$, all hydrographs are concave upwards. For $n=1$, the solution of eq (11) is an exponential function. All functions for $n\leq 1$ are asymptotic with the time axis. Hydrographs for $n\geq 1$ do not have this asymptote, but cut the $t-$axis for finite values (for $n=2$, $t_0=2Q_0/c$). For the range $2>n>1$, the hydrographs are power functions and concave upward, but have finite values for $t=0$ (are not asymptotic to the t axis). For $n<1$ the curves are hyperbolic functions, asymptotic and concave upward.

All outflow hydrographs for very small water levels above the bottom of outflow orifice become asymptotic, because their outflow functions change due to the change of type of flow or to some secondary effects. In closed-type outflow the function changes from $r=\frac{1}{2}$ to about $r=\frac{3}{2}$ as soon as the water level drops to near the upper edge of the outflow orifice, and changes further for smaller outflow heights due to surface tension and other secondary effects. The exponent r increases so that $n=m/r\leq 1$.

The shapes of outflow hydrographs depend thus on the ratio of exponents of storage and outflow functions, and they can be: hyperbolic $(n<1)$, exponential $(n=1)$, power function with exponents higher than 1 $(2>n>1)$, straight line $(n=2)$ and power function with exponents lower than 1 $(n>2)$. They show that the natural conditions can produce different outflow hydrographs on the same river, depending on the lake and channel characteristics and outflow shapes.

The free spillways of reservoirs, or free outflows of shallow lakes, where the range of storage fluctuation is small, $m=1.0$ to 1.3, and $r\geq\frac{3}{2}$, have $n\leq 1$, and all outflow hydrographs are hyperbolic asymptotic functions (fig. 6). For the higher range of levels of outflow, m is greater, and the outflow hydrographs are power functions whose graphs are convex upward or downward. The smallest value of n for closed-type outflow is $n=\frac{1}{0.5}=2$, so that all hydrographs are power functions convex upward.

In case eq (6a) is used instead of eq (6), the following general solution of eq (7) for $P=0$ is obtained:

$$t=\sum_{m=1}^{m}\frac{m}{C_m(m-r)}\left(Q_0^{\frac{m-r}{r}}-Q^{\frac{m-r}{r}}\right), \tag{14}$$

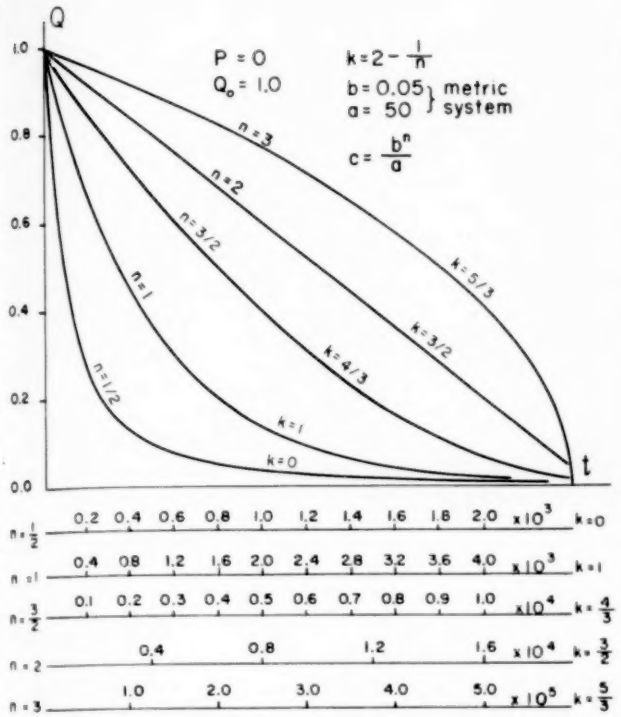

FIGURE 6. *Outflow hydrographs in their recession parts from Q_0 to zero, related to different values k and n, in the case where inflow is zero.*

which for the case $m \geq 3$ is more complicated than eqs (12) or (13), particularly since eq (14) gives t as a function of Q rather than the more convenient inverse.

If both types of flow occur simultaneously, eq (5b) has analytical solutions in a much more limited number of cases.

4.2. $P = P_o$ = Constant

In the case of constant or nearly constant inflow into the reservoir, with $P = P_o$, eq (7) becomes

$$y' + cP_0 y^2 - cy^k = 0, \tag{15}$$

and the separation of variables gives

$$\int \frac{dy}{P_0 y^2 - y^k} + ct = K \tag{16}$$

with K = integration constant.

The indicated integration of eq (16) cannot be carried out analytically for arbitrary k, contrary to the case of eq (11). Equation (16) has, however, analytical solutions for many rational numbers, which is useful, since it is possible to approximate k by a rational number of type g/h, where g and h are integers. Equation (16) can be written in the form of the binomial integral

$$\int y^{-2}(P_0 - y^{k-2})^{-1} dy + ct = K. \tag{17}$$

The binomial $y^{-2}(P_0 - y^{k-2})^{-1}$ is of the form

$$y^s(ay^r + b)^p \tag{18}$$

which can be reduced to an algebraic function in case any one of the three numbers

$$p, \quad \frac{s+1}{r}, \quad \text{and} \quad \frac{s+1}{r} + p$$

is an integer. As $p = -1$ is an integer, when k is a rational number, eq (16) can always be reduced to an algebraic function.

The most general solutions of the binomial integral, if the obtained algebraic functions are integrable, are in the form of sum of logarithmic, algebraic and arctan terms. In some cases the arctan term does not appear in the solution, as is the case also with the algebraic term, but the logarithmic term is always present. The solutions of eq (16) will be analyzed here for some values of k.

By substitution of a new variable u in eq (16),

$$u = y^{1/h} \cdot P_0^{\frac{1}{2h-g}}, \tag{19}$$

the following general binomial expression of eq (16) for $k = g/h$ is obtained:

$$hP_0^{\frac{g-h}{2h-g}} \int u^{h-g-1}(u^{2h-g} - 1)^{-1} du + ct = K \tag{20}$$

with $h-g-1$ and $2h-g$ integers.

For

$$f(u) = \int u^{h-g-1}(u^{2h-g} - 1)^{-1} du \tag{21}$$

the constant K for $t = 0$ $(u = u_0)$ is $K = hP_0^{(g-h)/(2h-g)} \times f(u_0)$, and the general solution has the form

$$t = R[f(u_0) - f(u)] \tag{22}$$

with $R = \frac{h}{c} P_0^{(g-h)/(2h-g)}$.

The following values and ranges of g and h are analyzed: (1) $g = 2h$; (2) $h \leq g < 2h$; (3) $g < h$; (4) $g = 0$, $h \neq 0$.

(1) $g = 2h$. For $k = 2$, eq (20) has no practical significance.

(2) $h \leq g < 2h$. The value $s = 2h - g$ is always a positive integer number, and $j = h - g - 1$ is always a negative integer number or zero. Equation (20) is integrable if $s = 2h - g \leq 4$ or $s = 6$, when the rational function can be separated in integrable terms, whatever the value of $j = h - g - 1$. The solution is

$$t = \frac{h}{c} P_0^{(h/s)-1} [f(u_0) - f(u)]. \tag{22a}$$

(a) $s=2h-g=1$, $n=h$.

$$f(u)=\ln\frac{u-1}{u}+\frac{1}{u}+\frac{1}{2u^2}+\ldots+\frac{1}{(n-1)u^{n-1}} \qquad (21a)$$

with $u=P_0Q^{-1}$.

This case covers the following values of n and k:

n: 1, 2, 3, 4, 5, 6, 7, etc.
k: 1, 3/2, 5/3, 7/4, 9/5, 11/6, 13/7, etc.

Some of these cases are given in table 2.

For $n=1$ ($k=1$) only the logarithmic part of eq (21a) is present and eq (22a) gives:

$$Q=P_0+(Q_0-P_0)e^{-ct} \qquad (21b)$$

(b) $s=2h-g=2$, $n=h/2$.

$$f(u)=\ln\frac{u-1}{u+1}+\frac{2}{u}+\frac{2}{2u^3}+\ldots+\frac{2}{2(n-1)u^{2(n-1)}} \qquad (21c)$$

with $u=P_0^{1/2}Q^{-1/2}$. This case covers the values:

n: 3/2, 5/2, 7/2, 9/2, etc.
k: 4/3, 8/5, 12/7, 16/9, etc.

Some of these are given in table 2.

TABLE 2. $P=P_0=Constant$: $t=R\,[f(u_0)-f(u)]$

n	n	k	R	u	$f(u)$
$\frac{1}{4}$	0.25	-2	$\frac{1}{c}P_0^{-\frac{3}{4}}$	$P_0^{\frac{1}{4}}Q^{-\frac{1}{4}}$	$\ln\frac{u-1}{u+1}+2\arctan u$
$\frac{1}{3}$	0.33	-1	$\frac{1}{c}P_0^{-\frac{2}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\ln\frac{(u-1)^2}{u^2+u+1}+4\arctan\left(\frac{2u+1}{3}\right)$
$\frac{1}{2}$	0.50	0	$\frac{1}{c}P_0^{-\frac{1}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}$
$\frac{2}{3}$	0.67	$\frac{1}{2}$	$\frac{2}{c}P_0^{-\frac{1}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\ln\frac{(u-1)^2}{u^2+u+1}-4\arctan\left(\frac{2u+1}{3}\right)$
$\frac{3}{4}$	0.75	$\frac{2}{3}$	$\frac{3}{c}P_0^{-\frac{1}{4}}$	$P_0^{\frac{1}{4}}Q^{-\frac{1}{4}}$	$\ln\frac{u-1}{u+1}-2\arctan u$
1	1.00	1	$\frac{1}{c}$	P_0Q^{-1}	$\ln\frac{u-1}{u}$; $Q=P_0+(Q_0-P_0)e^{-ct}$
$\frac{5}{4}$	1.25	$\frac{6}{5}$	$\frac{5}{c}P_0^{\frac{1}{4}}$	$P_0^{\frac{1}{4}}Q^{-\frac{1}{4}}$	$\ln\frac{u+1}{u-1}+\frac{4}{u}+2\arctan u$
$\frac{4}{3}$	1.33	$\frac{5}{4}$	$\frac{4}{c}P_0^{\frac{1}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\frac{1}{2}\ln\frac{(u-1)^2}{u^2+u+1}+\frac{3}{4}+2\arctan\left(\frac{2u+1}{3}\right)$
$\frac{3}{2}$	1.50	$\frac{4}{3}$	$\frac{3}{c}P_0^{\frac{1}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}+\frac{2}{u}$
$\frac{5}{3}$	1.67	$\frac{7}{5}$	$\frac{5}{c}P_0^{\frac{2}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\frac{1}{2}\ln\frac{(u-1)^2}{u^2+u+1}+\frac{3}{2u^2}-2\arctan\left(\frac{2u+1}{3}\right)$
2	2.00	$\frac{3}{2}$	$\frac{2}{c}P_0$	P_0Q^{-1}	$\ln\frac{u-1}{u}+\frac{1}{u}$
$\frac{7}{3}$	2.33	$\frac{11}{7}$	$\frac{7}{c}P_0^{\frac{4}{3}}$	$P_0^{\frac{1}{3}}Q^{-\frac{1}{3}}$	$\ln\frac{(u^3-1)^2}{u}+\frac{1}{3u^3}$
$\frac{5}{2}$	2.50	$\frac{8}{5}$	$\frac{5}{c}P_0^{\frac{5}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}+\frac{2}{u}+\frac{2}{3u^3}$
3	3.00	$\frac{5}{3}$	$\frac{3}{c}P_0^2$	P_0Q^{-1}	$\ln\frac{u-1}{u}+\frac{1}{u}+\frac{1}{2u^2}$
$\frac{7}{2}$	3.50	$\frac{12}{7}$	$\frac{7}{c}P_0^{\frac{5}{2}}$	$P_0^{\frac{1}{2}}Q^{-\frac{1}{2}}$	$\ln\frac{u-1}{u+1}+\frac{2}{u}+\frac{2}{3u^3}+\frac{2}{5u^5}$
4	4.00	$\frac{7}{4}$	$\frac{4}{c}P_0^3$	P_0Q^{-1}	$\ln\frac{u-1}{u}+\frac{1}{u}+\frac{1}{2u^2}+\frac{1}{3u^3}$

(c) $s=2h-g=3$, $n=h/3$. The solutions for three values of h and n ($h=4$, $n=4/3$; $h=5$, $n=5/3$; $h=7$, $n=7/3$) are given in table 2.

(d) $s=2h-g=4$, $n=h/4$. For $h=5$, $n=5/4$ the solution is given in table 2.

(3) $g<h$. The analytical integration can be carried out when $-4\leq 2h-g=s\leq 4$, or $g/2-2\leq h\leq g/2+2$. For $h>g/2+2$ and $h<g/2-2$ the exponent $s=2h-g$ in eq (20) is higher than 4, so that the algebraic function $u^{g-h+1}(u^{2h-g}-1)$ can not be separated into integrable components except in special cases. The solutions for $n=3/4$ ($k=2/3$), $n=2/3$ ($k=1/2$), $n=1/3$ ($k=-1$) and $n=1/4$ ($k=-2$) are given in table 2.

(4) $g=0$, $h=1$; $n=1/2$ ($k=0$). The solution is given in table 2. Figure 7 gives five curves for the values n: 1/2, 1, 3/2, 2 and 3, as in figure 6. All five are asymptotic to the value $P_0=0.20$, but for t nearer to zero they approach the curves of figure 6.

4.3. $P=f(t)$

The solution of eq (3) for four types of $P=f(t)$ can be obtained for special values of k, but the general analytical solution such as was obtained in the case of eq (13), or in many special cases of eq (22), is not possible.

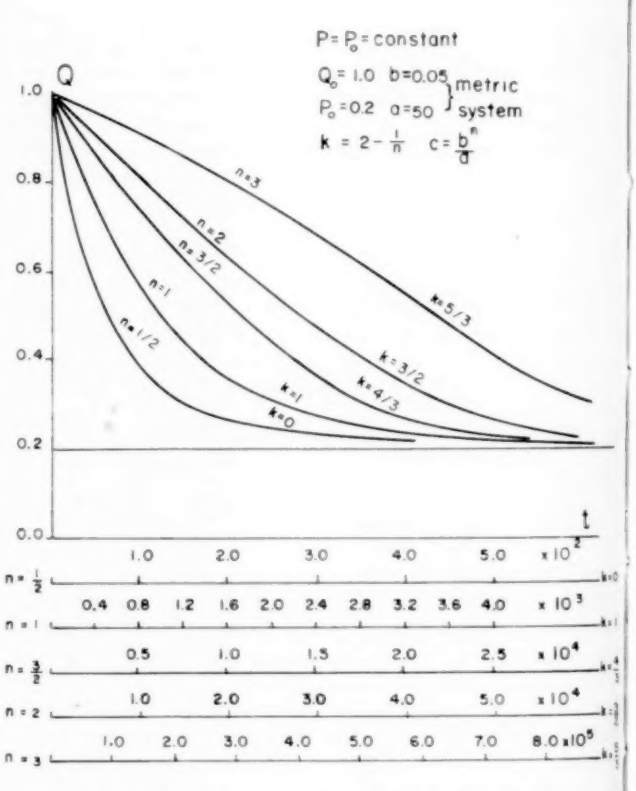

FIGURE 7. *Outflow hydrographs in their recession parts from Q_0 to inflow $P=P_0=$constant, related to different values k and n.*

a. $k=1$ $(n=1)$

Equation (7) becomes

$$y'+cPy^2-cy=0, \tag{23}$$

which is a Bernoulli equation, and has the general solution for $y=Q^{-1}$:

$$Q=e^{-ct}\left(K+c\int Pe^{ct}dt\right). \tag{24}$$

The solutions for four types of function $P=f(t)$ are given in table 3.

TABLE 3. $P=f(t)$

$k=1$, $n=1$, $s=$ *integer*

$P=f(t)$	$Q=f_2(t)$
$P=P_0-ft$	$\left(P_0-ft+\frac{f}{c}\right)+\left(Q_0-P_0-\frac{f}{c}\right)e^{-ct}$
$P=P_0t^s$	$\left[Q_0-\frac{P_0}{c^s}(-1)^s s!\right]e^{-ct}+\frac{P_0}{c^s}s!(-1)^s\left[1-\frac{ct}{1!}+\frac{c^2t^2}{2!}-\ldots+(-1)^s\frac{c^st^s}{s!}\right]$
$P=P_0e^{-ft}$	$\left(Q_0-\frac{cP_0}{c-f}\right)e^{-ct}+\left(\frac{cP_0}{c-f}\right)e^{-ft}$
$P_0=P_0t^se^{-ft}$	$\left[Q_0-(-1)^{s+1}\frac{s!cP_0}{(c-f)^{s+1}}\right]e^{-ct}-(-1)^{s+1}\frac{cP_0s!}{(c-f)^{s+1}}$ $[1-1!(c-f)t+2!(c-f)^2t^2-\ldots+s!(c-f)^st^s]e^{-ft}$

Such analytical solutions can be obtained under condition that the integration of

$$\int Pe^{ct}dt \tag{25}$$

in eq (23) can be carried out in closed form.

b. $k=0$ $\left(n=\frac{1}{2}\right)$

Equation (7) becomes

$$y'+cPy^2-c=0 \tag{26}$$

which is a special type of the Riccati equation, with $y=Q^{-1/2}$. As two particular integrals of eq (26) are known, viz, $y_0=Q_0^{-1/2}$ for $t=0$, and $Q=0$ or $Q=P_0$ for t infinite, the solution of eq (26) is possible. The substitution $Z=y-y_0$ gives

$$Z'+cPZ^2+2cPy_0Z+(y_0'+cPy_0^2-c)=0$$

and eq (26) becomes the Bernoulli equation

$$Z'+cPZ^2+2cPy_0Z=0 \tag{27}$$

with the solution

$$\frac{1}{y-y_0}=e^{2cy_0\int Pdt}\left(K+c\int Pe^{-2cy_0\int Pdt}dt\right) \tag{28}$$

The constant K is to be determined from the known value of Q for t infinite.

The solution for $P=P_0e^{-ft}$ is

$$Q=Q_0\left(\frac{e^{ge^{-ft}}-1}{e^{ge^{-ft}}+1}\right)^2 \tag{29}$$

where

$$g=2cP_0/f\sqrt{Q_0}.$$

The analytical integration is not possible for the three other P functions of table 3, because the integral $\int e^{at^s}dt$ can be obtained, in general, only in power series form.

This analysis shows that the analytical integration for simple expressions $P=f(t)$ can be done for $k=1$ and for $k=0$ in some cases.

5. Discussion

The analysis of the feasibility of fitting the given curves, as parts of differential equation for water storage, by mathematical expressions and its integration by analytical procedures in special cases, show some characteristics of outflow hydrographs.

The shape of outflow hydrographs from lakes and channel reaches is highly influenced by the ratio of exponents of storage function and outflow rating curve, both of the power type $y=sx^p$.

The analytical integration of the storage differential equation can be performed for any ratio of exponents in case of zero inflow, and for many ratios of exponents in case of constant inflow. The integration is, however, limited to a small number of ratios and inflow functions, when the inflow changes with time.

The usual procedure, to plot the recession part of river-flow hydrographs on semi-log paper and to fit it by a straight line, shows that in the majority of cases the fitting of a straight line is a rough approximation. The results of above analytical integration show that the straight line can be used for accurate fitting only if the ratio of exponents of storage function and outflow rating curve has a mean value near unity $(n=m/r=1)$ and for zero or constant inflow. This can be considered a special case.

Though the discharge hydrographs of underground water in a river basin can be (theoretically) fitted in many cases during the recession period of no water supply to the underground by an exponential function of type Q_0e^{-ft}, the water storage either in lakes or along river channels influences highly the shape of river hydrographs, so that it becomes less and less of the pure exponential type. The higher the ratio between the effective storage (storage which influences the outflow hydrograph) of lakes and channels to the total river flow during the period of recession flow, and the higher the departure of the ratio of two exponents from unity, the greater is their influence on the shape of river hydrographs and the larger is the departure from the original hydrograph of the recession flow from the underground.

As the ratio $n=m/r$ changes along the river channel, the accurate flood routing procedures must

take that fact into consideration.

The described analytical procedure is useful for the study of outflow hydrographs in case of rapid openings in water bodies (breaches of dams, rapid openings of gates and valves, breaches of channel walls or levees, etc.). The assumption of simple shape of openings and of simple inflow hydrographs (zero, constant inflow) is usual in case of dam breaches. As the study of dam breaches has to be made for many openings of quite different dimensions and shapes for different inflow discharges, the analytical integration as given above can have some advantages in the computation of hydrographs in comparison to the standard graphical or numerical procedures of integration.

The difficulties in fitting the background curves by tractable mathematical expressions and the difficulties of integrating analytically the storage differential equation, and especially in case the inflow changes with time, explain why the usual graphical and numerical procedures have taken so common a place in the engineering practice. Nevertheless, the analytical treatment of the storage differential equation can be useful for some problems, can serve for better understanding of relations, and can save efforts in problems, as, for instance, in case of outflow hydrographs out of dam breaches.

The assumptions usually taken for some breaches and for inflow discharge, and the accuracy of basic data in the routing of waves created downstream, justify the use of the procedure given for the fitting of mathematical expressions to the basic and given curves and the use also of the analytical integration of the storage differential equation for the computation of outflow hydrographs.

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Compressible Turbulent Boundary Layers With Heat Transfer and Pressure Gradient in Flow Direction[1]

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The best-known theoretical works on boundary-layer problems, especially in the case of compressible flow without or with heat transfer, are related to the laminar boundary layer, although the turbulent boundary layer is, in practice, often more interesting. The laminar boundary layer is more easily accessible to theoretical treatment because clearly defined relations exist between the viscosity μ and the shear stresses τ. In the turbulent case, empirical relations must be introduced. Therefore, attempts to get exact solutions are not worthwhile, while efforts to obtain approximate solutions, based for instance on the von Karman-Pohlhausen principle (Z. Angew. Math. u. Mech. **1**, 233 and **252**, 1921) of utilizing integral conditions, appear to be appropriate to this problem. In the last few years the accuracy of such approximate solutions for the incompressible case was noticeably improved by the application of a new energy integral condition in connection with a new empirical law for the dissipation in turbulent boundary layers, stated by J. Rotta (Ingr-Arch. **20**, 195, 1952) and E. Truckenbrodt (Ingr-Arch. **20**, 212, 1952). The empirical laws for dissipation and for turbulent wall friction, which are needed in the present approximate theory, are formulated on the basis of available measurements for incompressible flow. Generalization to the compressible flow with heat transfer is made from physical considerations. Calculated results agree satisfactorily with available experimental data. Some possibilities for improving as well as for simplifying the approximation theory are outlined.

1. Introduction

In principle, problems of compressible *laminar* boundary layers may be solved by using the partial differential equations for the velocity and temperature fields, and the thermodynamic relations between viscosity μ, density ρ, pressure p, and temperature T. Exact solutions for special cases have been worked out by several authors [5, 6, 7, 8].[2] Nevertheless, approximate theories of the von Karman-Pohlhausen type have been successful because of their easy adaptability to the most general case of flows with arbitrary pressure gradient in flow direction [9 through 16].

An example is the case of a compressible *turbulent* boundary layer, which is more important in practice because the controlling viscosity is no longer a material property, such as the molecular viscosity, μ; rather, it is like so-called "apparent viscosity," μ_a, which is essentially determined by the empirical laws of turbulent mixing motions. In view of the empirical character of these laws, attempts of exact solutions of Prandtl's boundary layer equations are indeed inappropriate. Hence, approximation theories of the von Karman-Pohlhausen type are indeed the only ones which are well adapted to the problem of calculating turbulent boundary layers.

The first solution was given by E. Gruschwitz [17] in 1931. He introduced a one-parameter family of empirical turbulent velocity profiles with parameter H, and, as the second parameter of the problem, the boundary-layer momentum-loss thickness δ_2. Gruschwitz then derived an empirical energy law. This was used together with the known von Karman integral condition for the momentum within the boundary layer, to determine the two parameters H and δ_2 as functions of x for a given pressure gradient.

The so-called Gruschwitz energy law could not be verified in a general manner and appears therefore somewhat doubtful. Moreover, it is restricted to the incompressible case. A definite improvement of the approximate theory for turbulent boundary layers was accomplished

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[2] Figures in brackets indicate the literature references at the end of this paper.

when several authors [9 through 15] independently (between 1935 and 1946) showed that an integral condition of the energy within the boundary layer may be derived from the Prandtl boundary-layer equation in the similar way as for the momentum. Some years later (1950) J. Rotta [3] derived from measurements the empirical law for the dissipation within the turbulent boundary layer, which was needed for the applications of the new integral energy condition. At about the same time the empirical law for the turbulent wall friction occurring in the momentum condition was improved by new accurate measurements of Ludwieg and Tillmann [18], such that it may be generally valid for positive and negative pressure gradients in flow direction.

Now, on the basis of the momentum and energy integral conditions on the one hand and the two empirical laws for the turbulent wall friction and dissipation on the other hand, a two-variable method approximate theory of boundary layer could be developed, similar to that formerly worked out by several authors for laminar boundary layers [9 through 16].

For the incompressible case, the integration method of E. Truckenbrodt [4] (1952) (similar to that developed by A. Walz [19] in 1941 for the laminar case) appears to be the simplest. However, applications to compressible turbulent boundary layer failed, because the semi-empirical laws for wall friction and dissipation could not be set up, since before that time measurements of turbulent boundary layers at high Mach numbers were not available in sufficient generality.

Basing his analysis upon simple physical facts about the turbulence mechanism the author was able to derive a generalization of the empirical laws for wall friction and dissipation to include compressibility, [26, 27].

The inclusion of the case of heat transfer, however, implies a new difficulty insofar as the integral energy condition derived from Prandtl's equation considers only an energy flux along the body surface but does not relate to an energy flux through the wall. In the latter general case we must start from the general differential equation for the temperature field. In addition it must be checked whether the above compressible considerations would include heat transfer at the wall.

As suggested by L. E. Kalikhman [13] for the case of laminar boundary layers the partial differential equation for the temperature field can be transformed by partial integrations into an infinite system of ordinary differential equations similar to that obtained by partial integration of Prandtl's boundary layer equation.

Hence, two infinite systems of ordinary differential equations with a twice infinite number of unknowns are available which appear to be completely equivalent to the two partial differential equations for the velocity and temperature field.

Some general properties of these systems, which had not yet been outlined in the literature, will be discussed.

As a first step of evaluation of the possibilities involved in these systems of equations, a "two-equation, two-unknown" method is investigated. This is about as simple as the known methods for the incompressible case [4, 19]. Nevertheless results obtained are of sufficient accuracy for engineering purposes.

The application of the new method is illustrated in some examples. Directions for improvement as well as for further simplifications are indicated.

2. Partial Integration of Prandtl's Boundary-Layer Equation for a Compressible Two-Dimensional Steady State Flow Along an Insulated Wall

2.1. Derivation of an Infinite System of Ordinary Differential Equations

As proposed by Leibenson [9], Golubew [10], Sutton [11], K. Weighardt [12], and E. Truckenbrodt [4], Prandtl's system of partial differential equations for a compressible two-dimensional flow along an insulated wall may be transformed into an infinite system of ordinary differential equations by partial integrations. We will give here a new derivation of this

system which is appropriate for our purposes.

Let x, y be the coordinates, the x-direction being taken everywhere tangent to the wall. Further u and v are the components of the velocity vector, respecting in the x and y directions, μ the viscosity coefficient, δ the density coefficient. Then Prandtl's boundary-layer equations may be written as:

$$\rho u \frac{\partial u}{\partial x}+\rho v \frac{\partial u}{\partial y}=\rho_\delta u_\delta \frac{du_\delta}{dx}+\frac{\partial\left(\mu\frac{\partial u}{\partial y}\right)}{\partial y}, \tag{1}$$

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \quad \text{(continuity equation)}. \tag{2}$$

Here the subscript δ denotes that the quantity is referred to the edge of the boundary layer. We note that this system is valid even for turbulent boundary layers, if the velocities u and v as well as ρ are considered to be the mean values over time and if μ is the effective value $\mu\equiv\mu_e$, which consists of the molecular viscosity μ, the apparent viscosity μ_a. Thus,

$$\mu_e=\mu+\mu_a\,. \tag{3}$$

For simplicity, however, we prefer to use in both cases, laminar and turbulent, the same symbols u, v, ρ, and μ, knowing that in the turbulent case we have $\mu\equiv\mu_e$.

First we multiply eq (1) by u^ν and add to it eq (2) multiplied by $u^{\nu+1}/(\nu+1)$.

Partial integration from $y=0$ to $y=\infty$ then yields after elementary transformations,

$$\frac{1}{\rho_\delta u_\delta^{\nu+2}}\frac{d}{dx}\left(\rho_\delta u_\delta^{\nu+2} f_\nu\right)+g_\nu\frac{du_\delta/dx}{u_\delta}+e_\nu=0 \qquad (\nu=0,1,2,\ldots,\infty).^3 \tag{4}$$

Herein the following notations are used:

$$e_\nu=(\nu+1)\int_0^{u_\delta}\left(\frac{u}{u_\delta}\right)^\nu\frac{\partial}{\partial u}\left(\frac{\tau}{\rho_\delta u_\delta^2}\right)du, \tag{5}$$

$$f_\nu=\int_0^\infty\left[\frac{\rho u}{\rho_\delta u_\delta}-\frac{\rho}{\rho_\delta}\left(\frac{u}{u_\delta}\right)^{\nu+2}\right]dy, \tag{6}$$

$$g_\nu=-(\nu+1)\int_0^\infty\left[\frac{\rho u}{\rho_\delta u_\delta}-\left(\frac{u}{u_\delta}\right)^\nu\right]dy, \tag{7}$$

$$\tau=\mu\frac{\partial u}{\partial y}\,. \tag{8}$$

From the structure of the integral terms (5) through (7) we see that for practical purposes it will not be necessary to extend the integration process beyond a certain distance $y=\delta$, for which we have $u=u_\delta$, $\rho=\rho_\delta$, $\tau=0$, because for $y>\delta$ all integrands became practically zero. The layer thickness δ is somewhat vague in view of the asymptotic behavior of $u(y)$ at the outer edge. For approximate theories, however, it is very useful and may be defined as wall distance y at which u reaches the value u_∞ with an accuracy better than 99 percent. Hence we put

$$0.99u_\infty<u_\delta<u_\infty. \tag{9}$$

2.2. Discussion of the Infinite System of Equations

Putting $\nu=0$ we obtain the usual momentum integral condition of von Karman-Pohlhausen, for $\nu=1$ we have Weighardt's energy integral condition. Only these two cases have been considered up to the present, but it is clear that all other remaining integral conditions of the

[3] ν may have also values between $-\infty$ and $+\infty$, even nonintegers as 0.1, 0.2, . . . and so on.

infinite system (4), the so-called higher momentum conditions ($\nu=2, 3, 4, \ldots, \infty$), would be important too, since only the complete system is equivalent to Prandtl's equations.

Let us first investigate the meaning of the equations of order ν, higher than unity.

First eq (4) is rewritten in a more convenient form. At the outer edge $y=\delta$ of the boundary layer, we may assume adiabatic conditions of state. According to the basic laws of gas dynamics we have between the density of ρ_δ, the velocity u_δ, and the Mach number $M_\delta=u_\delta/a_\delta$ (a_δ=local sound velocity) the following relations (see, for instance, R. Sauer [20]):

$$\frac{1}{\rho\delta}\frac{d\delta}{dx}=-\frac{1}{u\delta}\frac{du_\delta}{dx}M_\delta^2. \tag{10}$$

Substitution of (10) into (4) gives

$$f_\nu'+\frac{u_\delta'}{u_\delta}f_\nu\left[2+\nu-M_\delta^2+\frac{g_\nu}{f_\nu}\right]+e_\nu=0 \tag{11}$$

where the prime denotes differentiation with respect to x.

The integral terms e_ν, f_ν, g_ν represent the unknowns of the system. They contain the velocity and density distributions $u(x,y)/u_\delta(x)$ and $\rho(x,y)/\rho_\delta(x)$, respectively, which are indeed the real unknowns of the problem.

The only way to obtain a solution of the problem was to introduce approximations for u/u_δ and ρ/ρ_δ according to von Karman-Pohlhausen's idea, for instance by means of a one-parameter family of experimental curves. In the incompressible case ($\rho/\rho_\delta=1$) the problem was indeed solved in this way by taking two equations from system (11) (for $\nu=0$ and $\nu=1$),[4] using as unknowns the boundary layer thickness δ or one of the functions e_ν, f_ν, g_ν, and, as the second unknown, a shape parameter H of the velocity profile. In this case the ratios e_ν/δ, $f\nu/\delta$, $g\nu/\delta$, as well as e_ν/f_ν and g_ν/f_ν, are only functions of the shape parameter H.

The compressible case involves new variables ρ and T which are related by

$$\frac{\rho}{\rho_\delta}=\frac{T_\delta}{T}, \tag{13}$$

since $\partial p/\partial y=0$. Additional equations will be needed to make the problem determinate. This is done for the thermal boundary layer. While we will return to this problem in section 3 of this report, let us first discuss the principal properties of the infinite system (11).

The order number ν may for instance be increased toward ∞. In this case we have from equations (6) and (7),

$$[f_\nu(\nu+2-M_\delta^2)+g_\nu]_{\nu=\infty}=f_\infty(1-M_\delta^2), \tag{14}$$

and

$$f_\infty=\int_0^\delta\frac{\rho u}{\rho_\delta u_\delta}dy, \tag{15}$$

because

$$\left(\frac{u}{u_\delta}\right)_{\nu\to\infty}^{\nu+2}=\left(\frac{u}{u_\delta}\right)_{\nu\to\infty}^{\nu}=0, \quad \text{for } 0<\frac{u}{u_\delta}<1, \tag{16}$$

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[4] The original Pohlhausen method only used the equation $\nu=0$ (momentum equation) from this system. As the second equation, Pohlhausen introduced the so-called wall condition,

$$\left(\mu\frac{\partial^2 u}{\partial y^2}\right)_{y=0}=\frac{dp}{dx}, \tag{12}$$

which follows directly from Prandtl's eq (1) for the point $y=0$. It is obvious that this condition, valid for the wall point, would be doubtful if used for fixing the shape parameter. Indeed essential improvements of results were observed when this wall condition was neglected and replaced by the equation for $\nu=1$ (see, for instance, [16]).

whatever the functions u/u_δ and ρ/ρ_δ may be in detail.

The integral expression $(e_\nu)_{\nu\to\infty}$ is found to be zero as follows:

For $0<\frac{u}{u_\delta}<1$ it yields

$$\left[(1+\nu)\left(\frac{u}{u_\delta}\right)^\nu\right]_{\nu\to\infty}\to 0.$$

Since furthermore the shear stress τ and its derivative $\partial\tau/\partial y$ are vanishing at $y=\delta$ $(u=u_\delta)$ for arbitrary velocity profile u/u_δ, we have

$$(e_\nu)_{\nu\to\infty}\to 0. \tag{17}$$

Hence in the limiting case of $\nu=\infty$, system (11) reduces to

$$f'_\infty + f_\infty(1-M_\delta^2)\frac{u'_\delta}{u_\delta}=0\,. \tag{18}$$

This equation may be called the *mass-flow equation* and is nothing less than an averaged form of the continuity equation. With the aid of (10) it can immediately be integrated to

$$f_\infty\rho_\delta u_\delta=(\delta-\delta_1)\rho_\delta u_\delta=\text{const.} \tag{19}$$

where

$$\delta_1=\int_0^\delta\left(1-\frac{\rho u}{\rho_\delta u_\delta}\right)dy \tag{20}$$

is known as the displacement thickness. This relation has not previously been used in approximate boundary layer theories. It expresses that the mass flow between the stream lines $y=\delta$ and $y=\delta_1$ must be constant. We have not ascertained whether relation (19) will be very useful in practice. But it is clear that if the velocity and density profiles are once determined in the inner part of the boundary layer (thereby determining the value δ_1), the total boundary layer thickness δ is determined also by relation (19).

We have found that if the two-equation system $\nu=0$, $\nu=1$ with two unknowns is used, this relation is mostly violated. Nevertheless the essential properties of the boundary layer, for example, wall friction and separation behavior, appear to be correctly described.

From this consideration about the mass-flow equation as limiting case for $\nu=\infty$, we may learn that the higher the order ν, the less details of the velocity profile will be included in the corresponding equation, because the term e_ν, which contains the shearing stress (and with this the entire boundary layer effect), approaches zero, with $\nu\to\infty$.

It is therefore not surprising that equations corresponding to low order, for example the equation for $\nu=0$ and $\nu=1$, still give a very good description of the properties of the boundary layer if a rather physically reasonable family of velocity profiles is introduced into the equations. A good way in principle as suggested for the incompressible laminar boundary layers a long time ago by several authors [9,10,11,12], would be to restrict to a finite number of equations of the system, preferably to those of low order, say up to $\nu=5$ or 10, and to introduce a polynomial expression for the velocity profile

$$\frac{u}{u_\delta}=\frac{u}{u_\delta}\left[\frac{y}{\delta_2(x)},H_1(x),H_2(x),H_3(x)\ldots,\ H_\nu(x)\right] \tag{21}$$

with ν coefficients H. Then $\nu+1$ ordinary differential equations will allow the determination of the ν coefficients H_ν and δ_2 as functions of x. In this case no detailed assumption about the velocity profile, except some trivial boundary conditions, must be made.

It appears worthwhile to study this problem with new efforts, because its solution will be to a large extent equivalent to the solution of Prandtl's boundary layer equation, but needs no restrictions as to the prescribed pressure distribution $p(x)$, and will in principle be applicable to the more important case of turbulent boundary layers too. Since we want to include the

compressible case with heat transfer at the wall, we must simultaneously consider the temperature boundary layer.

The infinite system of equations derived from Prandtl's equation contains no terms which take into account an energy flux through the wall. Therefore, only the momentum equation $\nu=0$ and the mass-flow equation $\nu=\infty$ of this system can be used in connection with heat transfer problems. All other equations are worthless in this respect. The essential step to get out of this difficulty is to include the partial differential equation for the temperature field in this problem by transforming it into an infinite system of ordinary differential equations similar to that obtained from Prandtl's equation.

3. Derivation of Integral Conditions From the Partial Differential Equation for the Temperature Field

The general partial differential equation for the temperature field in a two-dimensional steady state flow may be written as follows, if the enthalpy $i=c_pT$ is introduced:

$$\rho u\frac{\partial i}{\partial x}+\rho v\frac{\partial i}{\partial y}=u\frac{dp}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial}{\partial y}\left(\frac{\lambda}{c_p}\frac{\partial i}{\partial y}\right). \tag{22}$$

Here λ denotes the heat conductivity (kcal/m sec ^{0} K), c_p the specific heat at constant pressure, u, v must satisfy the continuity equation

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0. \tag{23}$$

Now, if we multiply eq (22) by i, and eq (23) by $i^{\nu+1}/\nu+1$ and add the multiplied equations, we obtain

$$\frac{\partial(\rho u i^{\nu+1})}{\partial x}+\frac{\partial(\rho v i^{\nu+1})}{\partial y}=(\nu+1)\left\{-u\rho_\delta u_\delta\frac{du_\delta}{dx}i^\nu+\mu\left(\frac{\partial u}{\partial y}\right)^2 i^\nu+\frac{\partial}{\partial y}\left(\frac{\lambda}{c_p}\frac{\partial i}{\partial y}\right)i^\nu\right\}. \tag{24}$$

By partial integration of eq (24) with respect to y from $y=0$ to $y=\infty$ and by substitution for v from eq (23) we have

$$\frac{d}{dx}\left[\int_0^\infty(\rho u i^{\nu+1}-\rho_\delta u_\delta i_\delta^{\nu+1})dy\right]+\frac{d}{dx}\left[(\rho_\delta u_\delta i_\delta^{\nu+1})\delta\right]-i_\delta^{\nu+1}\frac{d}{dx}\left[\int_0^\infty(\rho u-\rho_\delta u_\delta)dy\right]-i_\delta^{\nu+1}\frac{d}{dx}\left[\rho_\delta u_\delta\delta\right]$$

$$=(\nu+1)\left\{-\rho_\delta u_\delta\frac{du_\delta}{dx}\int_0^\infty(ui^\nu-u_\delta i_\delta^\nu)dy-\rho_\delta u_\delta\frac{du_\delta}{dx}u_\delta i_\delta^\nu\delta+\int_0^\delta i^\nu d\left[\frac{\lambda}{c_p}\frac{\partial i}{\partial y}\right]+\int_0^\infty\mu i^\nu\left(\frac{\partial u}{\partial i}\right)^2 dy\right\}. \tag{25}$$

The terms with the factor δ vanish because of the following known relation in gas dynamics:

$$\frac{1}{i_\delta}\frac{di_\delta}{dx}+\frac{u_\delta^2}{i_\delta}\frac{1}{u_\delta}\frac{du_\delta}{dx}=0, \tag{26}$$

which follows after a short intermediate calculation from

$$\frac{u_\delta^2}{i_\delta}=(k-1)M_\delta^2,\qquad k=\frac{c_p}{c_v}, \tag{27}$$

and

$$M_\delta^2=\frac{2}{k+1}\frac{M_\delta^{*2}}{1-\frac{k-1}{k-1}M_\delta^{*2}};\qquad M_\delta^*=\frac{u_\delta}{a^*};\qquad a^*=\text{critical sound velocity} \tag{28}$$

$$\frac{1}{M_\delta}\frac{dM_\delta}{dx}=\frac{1}{u_\delta}\frac{du_\delta}{dx}\left(1+\frac{k-1}{2}M_\delta^2\right). \tag{29}$$

Using notations for the integral terms:

$$\int_0^\infty \frac{\rho u}{\rho_\delta u_\delta}\left[\left(1-\frac{\rho_\delta}{\rho}\right)^{\nu+1}\right]dy \approx \int_0^\delta \frac{\rho u}{\rho_\delta u_\delta}\left[1-\left(\frac{\rho_\delta}{\rho}\right)^{\nu+1}\right]dy=\phi_\nu \tag{30}$$

(∞ can be replaced by δ, because the integrand is almost vanishing for $y \geqq \delta$)

$$\int_0^1 \left(\frac{\rho_\delta}{\rho}\right)^\nu \frac{\tau}{\tau_w}\,d\left(\frac{u}{u_\delta}\right)=\epsilon_\nu, \tag{31}$$

$$\int_0^{u_\delta}\left(\frac{\rho_\delta}{\rho}\right)^\nu \frac{\partial}{\partial u}\left[\frac{\tau}{\tau_w}\frac{\partial(T/T_\delta)}{\partial(u/u_\delta)}\right]du=\gamma_\nu, \tag{32}$$

with

$$\tau=\mu\frac{\partial u}{\partial y},\quad \tau_w=\left[\mu\frac{\partial u}{\partial y}\right]_{y=0},\quad \frac{\mu c_p}{\lambda}=P_r=\text{constant Prandtl number}, \tag{33}$$

and

$$\frac{T}{T_\delta}=\frac{i}{i_\delta}=\frac{\rho_\delta}{\rho}=\left(\text{because }\frac{\partial p}{\partial y}=0\right). \tag{34}$$

equation (25) may be rewritten as follows:

$$\frac{d\phi_\nu}{dx}+\phi_\nu\frac{du_\delta}{dx}\frac{1-M_\delta^2}{u_\delta}=-(\nu+1)\frac{k-1}{2}M_\delta^2\frac{\tau_w}{\rho_\delta u_\delta^2}\left\{2\epsilon_\nu+\frac{\gamma_\nu}{P_r\dfrac{k-1}{2}M_\delta^2}\right\}. \tag{35}$$

ϕ_ν, ϵ_ν and γ_ν (with ν as an arbitrary number) are an infinite number of unknowns in this infinite system of ordinary differential equations. The real unknowns, however, are the distributions $u(x,y)/u_\delta(x)$ and $T(x,y)/T_\delta(x)$ which we are trying to determine by means of the infinite number of integral conditions as expressed in the system (35). Now, before going to the solution, let us transform it into a somewhat more convenient form. First a new variable is introduced:

$$\eta_\nu=\frac{\phi_\nu}{\delta_2(k-1)M_\delta^2}, \tag{36}$$

where δ_2 is the displacement thickness, defined by

$$\delta_2=\int_0^\delta \frac{\rho u}{\rho_\delta u_\delta}\left(1-\frac{u}{u_\delta}\right)dy=f_0 \qquad \text{(see (11))}. \tag{37}$$

It is to be remembered that η_ν is finite when $M_\delta \to 0$, while ϕ_ν has the disadvantage of becoming zero when $M_\delta \to 0$. Furthermore,

$$\Phi=1+(k-1)M_\delta^2-\frac{\delta_1}{\delta_2},\qquad \delta_1=\int_0^\delta\left(1-\frac{\rho u}{\rho_\delta u_\delta}\right)dy, \tag{38}$$

$$\Psi_\nu=(\nu+1)\frac{\tau_w}{\rho_\delta u_\delta^2}\left\{\epsilon_\nu+\frac{\gamma_\nu}{P_r(k-1)M_\delta^2}+\frac{\eta_\nu}{\nu+1}\right\}. \tag{39}$$

With those notations (35) can be rewritten as follows:

$$\boxed{\frac{d\eta_\nu}{dx}+\eta_\nu\Phi\frac{d'u_\delta/dx}{u_\delta}+\frac{\Psi_\nu}{\delta^2}=0} \qquad (\nu=0,1,2,\ldots,\infty). \tag{40}$$

In principle a possible way to solve the problem of determining the distribution $u(x,y)/u_\delta(x)$ and $\rho(x,y)/\rho_\delta(x)$ is to use polynomial expressions which fulfill the boundary conditions of the velocity and thermal boundary layer, by choosing j coefficients which satisfy j equations of the system (40). It is obvious that in the present case the procedure is more complicated than in the case of system (11). However, it may be possible to overcome by taking the solution from the exact theory for the temperature profile T/T_δ or the density profile $\rho/\rho_\delta = T_\delta/T$. Although the solution is exactly valid only for the special case of the flat plate ($dp/dx=0$), it may be considered as a good approximation even for the more general case with $dp/dx \neq 0$ (see for instance H. Schlichting [21, p. 282]). An essential property of this solution for T/T_δ is that it depends not upon x but only upon u/u_δ with the Mach numbers M_δ and the heat transfer coefficient c_T as parameters.

With reference to a paper of von Driest [23] this solution for T/T_δ for the case $dp/dx=0$ can be written as

$$\frac{T}{T_\delta} = 1 + c_T(f_1 - 1) + r\frac{k-1}{2}M_\delta^2(1-f_2) \tag{41}$$

with

$$f_1\left(\frac{u}{u_\delta}, P_r\right) \approx \frac{u}{u_\delta}, \quad f_2\left(\frac{u}{u_\delta}, P_r\right) \approx \left(\frac{u}{u_\delta}\right)^2 \quad \text{for } 0.8 < P_r < 1, \tag{42}$$

$$r = \frac{T_r - T_\delta}{T_0 - T_\delta} = \text{recovery factor}, \tag{43}$$

$$c_T = \frac{T_r - T_w}{T_\delta} = \text{heat transfer parameter}. \tag{44}$$

Thus the temperature and density profiles occurring in the integral conditions (36) to (39) are uniquely correlated to the velocity profiles u/u_δ, which now are the only unknown functions in the problem. So far all considerations and derivations are valid both for the laminar and turbulent case. For the solution of the laminar case, we have only to introduce an expression for the velocity profiles. The viscosity, as principal parameter of all boundary layer theories is known to be dependent only upon the temperature T, for instance according to Sutherland's law.

$$\frac{\mu}{\mu_1} = \frac{T_1 + C}{T + C}\left(\frac{T}{T_1}\right)^{3/2} \tag{45}$$

(with $C=113°\text{K}$ for air, T_1 being a reference temperature) or to its more convenient approximation

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^{\omega} \tag{46}$$

with

$$\omega = \frac{3}{2} - \frac{1}{1 + C/T_1} \tag{47}$$

($\omega \approx 0.80$ for temperatures T between 233° and 373°K, $\omega = 0.5$ for very high temperatures, $\omega = 1.5$ for very small temperatures near the zero point).

In the case of turbulent boundary layers which are mainly considered in this report, an empirical relation for the effective viscosity (as the sum of the apparent and molecular viscosity) must be introduced besides the assumption for the velocity profiles. Before this is done, a simplification of the systems (11) and (40) will now be considered.

4. Special Case of a Two-Equation System

4.1 General Considerations

As a first step of evaluation of the possibilities involved in systems (11) and (40) we will make a so-called one-parametric representation for u/u_δ, viz:

$$\frac{u}{u_\delta}=\frac{u}{u_\delta}\left(\frac{y}{\delta(x)},\quad H(x)\right), \tag{48}$$

where $H(x)$ is the form parameter of the velocity profile (as one unknown of the problem) and $\delta(x)$ is the boundary layer thickness (as second unkown). Therefore, to be correct, the relations (48) should be denoted as a two-parametric one.

It must be pointed out that $\delta(x)$ may be replaced by another characteristic length of the boundary layer, preferably by the momentum-loss thickness $\delta_2(x)$, thus avoiding any assumption as to the point where u/u_δ reaches the value 1.

Since we have now decided to use only two unknowns, $H(x)$ and $\delta_2(x)$, we need only two equations besides the empirical law for effective viscosity.

We choose the simplest equations (namely those for $\nu=0$) from both the system (11) based upon Prandtl's equation (1) and the system (40) based upon the general temperature field equation (2).[5]

These equations are:

a. The momentum integral condition from system (11). This is the only equation of the system (11) which, besides the mass-flow equation, is valid even in the case with heat transfer at the wall.

$$\boxed{f_0'+f_0\left[\frac{g_0}{f_0}+2-M_\delta^2\right]\frac{u_\delta'}{u_\delta}+e_0=0}, \tag{49}$$

with

$$e_0=-\frac{\tau_w}{\rho_\delta u_\delta^2}, \tag{50}$$

$$f_0=\int_0^\delta\frac{\rho u}{\rho_\delta u_\delta}\left[1-\frac{u}{u_\delta}\right]dy=\delta_2, \tag{51}$$

$$g_0=\int_0^\delta\left(1-\frac{\rho u}{\rho_\delta u_\delta}\right)dy=\delta_1. \tag{52}$$

b. The energy integral condition from system (40)

$$\boxed{\eta_0'+\eta_0\Phi\frac{u_\delta'}{u_\delta}+\frac{\Psi_0}{\delta_2}=0} \tag{53}$$

with

$$\eta_0=-\frac{1}{(k-1)\,M_\delta^2}\frac{\int_0^\delta\frac{u}{u_\delta}\left(1-\frac{\rho}{\rho_\delta}\right)dy}{\delta_2}, \tag{54}$$

$$\Phi=1+(k+1)\,M_\delta^2-\frac{\delta_1}{\delta_2}\qquad\text{(independent of }\nu\text{)}, \tag{55}$$

$$\Psi_0=\frac{\tau_w}{\rho_\delta u_\delta^2}\left\{\int_0^1\frac{\tau}{\tau_w}\,d\left(\frac{u}{u_\delta}\right)+\eta_0-\frac{1}{2}\frac{\theta}{s}\right\}, \tag{56}$$

[5] The value of the simple mass-flow equation (19), which may also be obtained from system (40) by putting $\nu=-\infty$, appears to be limited, but has not yet been studied in detail.

$$\theta=\frac{c_T}{\frac{k-1}{2}M_\delta^2}=\frac{T_r-T_w}{T_\delta}\frac{1}{\frac{k-1}{2}M_\delta^2}, \tag{57}$$

$$s\approx P_r\int_0^1\left(\frac{\tau}{\tau_w}\right)^{P_r-1}d\left(\frac{u}{u_\delta}\right)\approx P_r \qquad \text{(for } P_r=0,8 \text{ to } 1)\ . \tag{58}$$

For the practical application of the two systems (49) and (53) it was found to be very helpful to evaluate once for all the universal functions δ_1, δ_2, η_0, Φ, and Ψ_0 for the interesting range of the parameters H, M_δ, and θ (or c_T). Such an evaluation will be based upon a certain physically reasonable assumption about the velocity profile, as represented by (48). These assumptions for u/u_δ must of course be different in the laminar and turbulent cases. In the turbulent case, the empirical laws for the wall friction τ_w and the dissipation integral $\int_0^{u_\delta}\tau\partial u$ must also be introduced. The evaluation for the laminar boundary layer contains no difficulties, and will be dealt with in a separate report. Only the more complicated case of the turbulent boundary layer, which needs some special considerations will be treated in this report.

4.2. Two-Equation Method for Turbulent Boundary Layers

We now consider: a) A suitable expression for empirical turbulent velocity profiles, b) an empirical law for the wall friction in a compressible turbulent boundary layer with heat transfer, c) an empirical law for the dissipation in a compressible turbulent boundary layer with heat transfer, and d) a method of solution of the system of equations (49) and (53).

a. The Turbulent Velocity Profile

Detailed investigations of many authors [23, 24, 3] have shown that the turbulent boundary layer can be divided into several (usually into two) regions with respect to the direction normal to wall, in each one of which different laws for the influence of the outer pressure distribution and the Reynolds number are valid. Thus a general representation for the velocity profile requires a somewhat complicated formula. If only mean properties (integral quantities) of the boundary layer are considered and put into correlation as in our case, it will be sufficient to have an expression which contains less details but includes the whole range of the boundary.

Such a formula, which is proved valid even at high Mach numbers, is

$$\frac{u}{u_\delta}=\frac{y^{k(H)}}{\delta} \tag{59}$$

where k is the form parameter with values approximately 0.1 and 0.7, and is a unique function of any H to be chosen.

A disadvantage of (59) is of course that at the wall $y=0$ the value $(\partial u/\partial y)_{y=0}$ is infinite for all $0<k<1$ and that furthermore, at $y/\delta=1$, $\partial u/\partial y$ does not vanish as is physically required. But we have shown in previous publication [26, 27] that these facts do not essentially influence the integral terms of the approximate theory, since the wall-friction law and the relation between the velocity profile u/u_δ and the shear stress profile τ/τ_w are separately introduced as semi-empirical laws.

As an example of the suitability of this power law even for Mach numbers up to about 7 and with heat transfer at the wall, we refer to figure 1 which is drawn from experiments of R. E. Wilson [25]. This simple power law is very adequate to the purpose of this approximate theory with regard to the possibility of closed evaluation of the integral terms. Figure 2 shows the order of magnitude of the errors in the ratio $(\delta_3/\delta_2)_i$ plotted against $(\delta_1/\delta_2)_i$,

$$\left((\delta_3)_i=\int_0^\delta\frac{u}{u_\delta}\left[1-\left(\frac{u}{u_\delta}\right)^2\right]dy\right),$$

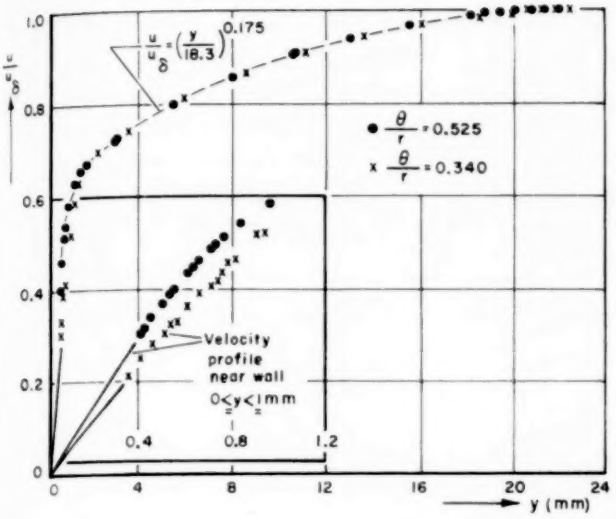


FIGURE 1. *Measured turbulent velocity profiles at a Mach number $M_\delta = 6.8$ and cooled wall $\left(\frac{\theta}{r} = 0.340\ (\times)$ and $0.525\ (\bullet)\right)$, from R. E. Wilson [25].*

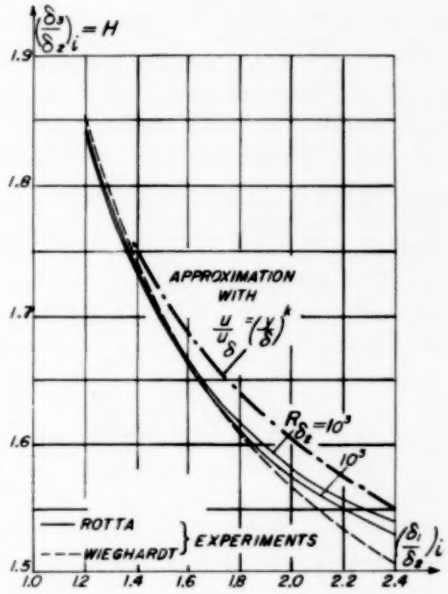


FIGURE 2. *Ratio $(\delta_3/\delta_2) = H$ plotted against the ratio $(\delta_1/\delta_2)_i$ for turbulent velocity profiles.*

Comparison of experiments (according to J. Rotta and K. Weighardt) with approximation power law $\frac{u}{u_\delta} = \left(\frac{y}{\delta}\right)^{k(H)}$.

caused by the approximation of the experimental velocity profiles with the power law calculated for the incompressible case. We see that the maximum error is about 1 percent. The neglect of a certain influence of the Reynolds number on the velocity profile causes errors less than 1 percent. Therefore, the use of the power law (59) for our approximate theory appears to be justified.

Some improvement at the outer edge region of the boundary layer can be reached without much complication by taking the following form:

$$\frac{u}{u_\delta} = \tanh\left[b \log\left(1 + ay\right)\right], \tag{60}$$

where a and b are two form-parameters, comparable to H_1 and H_2. a may be used for the purposes of the adjustment of scales in the one-parametric theory, and the adjustment of wall friction in the two-parametric theory. The relation (60) avoids the use of δ which is physically vague, and yields automatically an asymptotic behavior near the edge of the boundary layer. At the same time it prevents an infinite slope $\partial u/\partial y$ at $y = 0$.

Investigations of the practical importance of expressions (60) (which of course, excludes the possibility of closed integrations) are in progress.

b. The Wall Friction Law

For this we may refer to the author's publications dealing with the case of compressible turbulent boundary layers without heat transfer [26, 27]. We now have only to investigate to what extent the relations found there will be influenced by heat transfer at the wall.

As a summary of these earlier investigations we can state that the laws for turbulent mixing motion found for the incompressible case, are valid in the compressible case, provided that the fluctuation velocity is less than the sound velocity, in other words, if the velocity fluctuations are not accompanied by density fluctuations. Now, since the turbulent fluctuation velocity in boundary layers is found to be small, say 10 percent or 20 percent of the main velocity u_δ

at the outer edge of the boundary layer, $M_\delta = \frac{u_\delta}{a_\delta}$ may be increased to about 10 or 5, respectively, until the fluctation motion becomes compressible in character. Therefore we conclude that up to about Mach number $M=10$ or 5, respectively, the laws for incompressible fluctuation may be considered as valid to a first approximation. Only the flow medium properties μ and ρ are changed. These, however, are taken into account by a suitably chosen Reynolds number.

As to the effect of heat transfer at the wall, this will change only the flow medium properties and therefore can be included in the compressibility effect and in the Reynolds number, which itself plays a decisive role in the wall friction and dissipation law.[6]

The momentum law ($\nu=0$ from system (11)) contains the term $\tau_w/\rho_\delta u_\delta^2$ which is the ratio of the wall friction τ_w to twice the stagnation pressure. This ratio can be rewritten as follows:

$$\frac{\tau_w}{\rho_\delta u_\delta^2} = \frac{\mu_w \left(\frac{\partial u}{\partial y}\right)_w}{\rho_\delta u_\delta^2} = \frac{\left[\frac{\partial (u/u_\delta)}{\partial y/\delta_2}\right]_w}{\frac{\rho_\delta u_\delta \delta_2}{\mu_w}} = \frac{\left[\frac{\partial (u/u_\delta)}{\partial (y/(\delta_2)_i)}\right]_w}{R_{\delta_2}} \cdot \frac{\delta_2}{(\delta_2)_i}(H, M_\delta, \theta), \qquad (\delta_2)_i = \int_o^\delta \frac{u}{u_\delta}\left(1-\frac{u}{u_\delta}\right) dy. \quad (61)$$

A definition of a Reynolds number

$$R_{\delta_2} = \frac{\rho_\delta u_\delta \delta_2}{\mu_w} \qquad (62)$$

is suggested which contains the viscosity μ_w at the wall and, by means of the integral term δ_2, an averaged inertia term $\rho_\delta u_\delta \delta_2$. Indeed the momentum integral condition (49) brings into relationship the *wall* friction and the averaged inertia forces. The pressure forces are not of interest in this connection. Any other definition of the Reynolds number appears to be not suitable in the present problem. In the incompressible case these detailed considerations are of course not necessary.

In the numerator of eq (61) we have the dimensionless slope of $u(y)$ at $y=0$. As scale for the length y the value $(\delta_2)_{\text{incompr.}} = (\delta_2)_i$ as a wholly geometrical quantity, thus concentrating all influences of Mach number and heat transfer in the universal function $\delta_2/(\delta_2)_i$. This function is theoretically given, when expressions for u/u_δ and ρ/ρ_δ, (48) and (41), are fixed. Next the following term is considered:

$$\left[\frac{\partial (u/u_\delta)}{\partial (y/(\delta_2)_i}\right]_w \frac{1}{R_{\delta_2}} = \left(\frac{\tau_w}{\rho_\delta u_\delta^2}\right)_i \qquad (63)$$

of eq (61) as the wall friction law for incompressible flow. For turbulent boundary layers the slope $[\partial(u/u_\delta)/\partial y/\delta_2)_i]_w$ depends itself on the Reynolds number $R_{\delta 2}$ in such a manner that

$$\frac{\left[\frac{\partial (u/u_\delta)}{(y/(\delta_2)_i}\right]_w}{R_{\delta_2}} = \frac{\alpha(H)}{R_{\delta_2}^n} = \left(\frac{\tau_w}{\rho_\delta u_\delta^2}\right)_i \qquad (64)$$

with $n=0.268$ (according to Ludwieg-Tillman [18]) and

$$\alpha(H) = 0.123 \cdot 10^{-0.678(\delta_1/\delta_2)_i} \qquad (65)$$

as an empirical function, which expresses the influence of shape of u/u_δ.
Therefore, we can finally write

[6] A certain confirmation for these considerations may be seen in figure 1, which shows that the velocity distribution and therefore the related shear stress distribution τ/τ_w too, is the same as in the incompressible case.

$$\boxed{\frac{\tau_w}{\rho_\delta u_\delta^2}=\frac{\alpha(H)}{R_{\delta_2}^{0.268}}\cdot\frac{\delta_2}{(\delta_2)_i}(H,M_\delta,\theta)}. \tag{66}$$

c. The Dissipation Law

A generalization similar to that found for the wall friction can be derived for the dissipation in a compressible turbulent flow with heat transfer. The energy integral condition (53) contains the term

$$\int_0^{u_\delta}\tau du=D\ (\text{dissipation}). \tag{67}$$

Written in dimensionless form,

$$\frac{D}{\rho_\delta u_\delta^3}=\frac{\tau_w}{\rho_\delta u_\delta^2}\int_0^1\frac{\tau}{\tau_w}d\left(\frac{u}{u_\delta}\right)=\left[\left(\frac{\tau_w}{\rho_\delta u_\delta^2}\right)_i\int_0^1\frac{\tau}{\tau_w}d\left(\frac{u}{u_\delta}\right)\right]\frac{\delta_2}{(\delta_2)_i}. \tag{68}$$

The term $(\tau_w/\rho_\delta u_\delta^2)_i$ is given with eq (64). The remaining integral term is found empirically by T. Rotta [3] and E. Truckenbrodt [4] to be

$$\int_0^1\frac{\tau}{\tau_w}d\left(\frac{u}{u_\delta}\right)\approx\frac{\beta(H)}{\alpha(H)}R_{\delta_2}^{0.1}, \tag{69}$$

so that it finally yields

$$\boxed{\frac{D}{\rho_\delta u_\delta^3}=\left[\frac{\alpha(H)}{R_{\delta_2}^{0.168}}\frac{\beta(H)}{\alpha(H)}R_{\delta_2}^{0.1}\right]\frac{\delta_2}{(\delta_2)_i}=\frac{\beta(H)}{R_{\delta_2}^{0.168}}\frac{\delta_2}{(\delta_2)_i}} \tag{70}$$

with

$$\left(\frac{D}{\rho_\delta u_\delta^3}\right)_i=\frac{\beta(H)}{R_{\delta_2}^{0.168}}\approx\frac{0.0052}{R_{\delta_2}^{0.168}} \tag{71}$$

as the part which represents the dissipation law for the incompressible flow. To be quite correct, a detailed consideration must be made on the definition of the Reynolds number used in eq (69). By definition the dissipation integral expresses the amount of energy dissipated by the viscosity into heat (with the intermediary of turbulence). The related Reynolds number should be more exactly defined with a *mean* viscosity $\bar{\mu}$ instead of μ_w, because the energy transfer by viscosity occurs within the whole boundary layer, although mostly near the wall.

Therefore eq (69) should correctly be written as

$$\int_0^1\frac{\tau}{\tau_w}d\left(\frac{u}{u_\delta}\right)=\frac{\beta(H)}{\alpha(H)}\left(\frac{\mu_w}{\bar{\mu}}\right)^{0.1}R_{\delta_2}^{0.1} \tag{72}$$

when the definition (62) for the Reynolds number is used. This correction, however, was proved to be in general so small that it is not essential in the scope of this approximate theory.

d. Simplified Representation for the Wall Friction and Dissipation Law

The expressions for the incompressible wall friction and for the dissipation law were given by E. Truckenbrodt [4], who evaluated some semiempirical relations found by T. Rotta [3].

The results of the evaluation of E. Truckenbrodt are given in his two figures 3 and 4 which we have reproduced in the present figures 3 and 4. These have been presented in order to investigate whether or not it will be possible to find a new suitable fitting for these

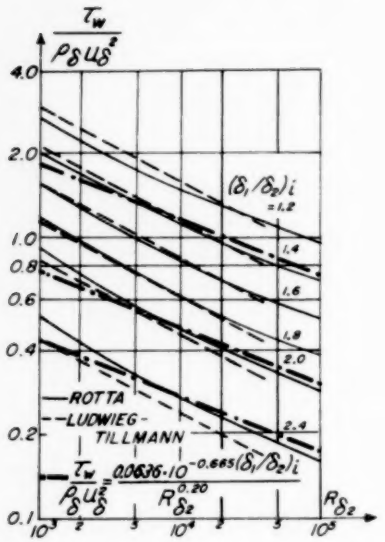


FIGURE 3. *Wall friction law for incompressible turbulent boundary layers.*

Thick line: evaluation of experiments of J. Rotta-Truckenbrod.
Dashed line: evaluation of experiments of Ludwieg-Tillmann.
Dashed-dotted line: new approximation with 0.2 power law.

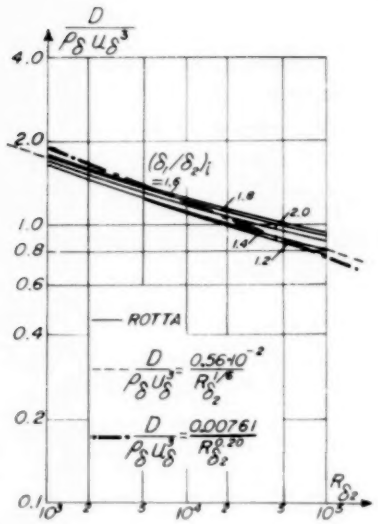


FIGURE 4. *Dissipation law for incompressible turbulent boundary layers.*

Thick lines: evaluation of J. Rotta-Truckenbrodt.
Dashed line: approximation proposed by Truckenbrodt.
Dashed-dotted line: new approximation with 0.2 power law.

two empirical laws which permits the use (for the sake of simplification) of only one power of the Reynolds number instead of two (0.268 and 0.168).

We find that if we take the power 0.2 in both cases the fitting of the experimental result in the most important range between $R_{\delta 2}=10^3$ and 10^5 will be rather satisfactory. The laws (66) and (70) may then be written as

$$\frac{\tau_w}{\rho_\delta u_\delta^2} = \frac{0.0636 \cdot 10^{-0.665(\delta_1/\delta_2)_i}}{R_{\delta_2}^{0.20}} \cdot \frac{\delta_2}{(\delta_2)_i} \tag{73}$$

$$\frac{D}{\rho_\delta u_\delta^3} = \frac{0.00761}{R_{\delta_2}^{0.20}} \cdot \frac{\delta_2}{(\delta_2)_i} \tag{74}$$

This simplification prevents the universal function Ψ_0 of equation (56), from being dependent upon $R_{\delta 2}$ besides H, M_δ and θ. Thus an important simplification in the representation of this function seems possible without noticable loss of accuracy. Detailed investigations on the reliability of this simplification are now in progress.

This simplification will be important if the use of higher momentum equations ($\nu=2, 3, 4, \ldots$) is made. In this case all empirical parts of the wall friction and dissipation law are concentrated in the term $\tau_w/\rho_\delta u_\delta^2$, and the integrals of type

$$\int_0^1 \left(\frac{\rho_\delta}{\rho}\right)^\nu \frac{\tau}{\tau_w}\, d\left(\frac{u}{u_\delta}\right) \quad \text{and} \quad \int_0^{u_\delta} \left(\frac{\rho_\delta}{\rho}\right)^\nu \frac{\partial}{\partial u}\left[\frac{\tau}{\tau_w}\frac{\partial(T/T_\delta)}{\partial(u/u_\delta)}\right] du$$

are dependent only upon the shape parameter H, the Mach number M_δ, and the heat transfer parameter θ. Therefore they can be evaluated, when the family of velocity profiles is chosen.

4.3. Evaluation of the Universal Functions of the Two-Equation Method for Laminar and Turbulent Boundary Layers

The evaluation of the universal functions occurring in the two-equation method as described in section 4 has already been made in the author's publications [26, 27] for the case of an insulating wall.

The evaluation of these functions for the case with heat transfer, as suggested by the author, is now under way at the DVL Institute for Applied Mathematics and Mechanics, Freiburg/Br, by considering first the turbulent boundary layer.

A preliminary evaluation for this case with slide rule accuracy has been available for about one year and was used for calculating some illustrating examples which will be discussed in the following section.

This preliminary evaluation showed that the effect of heat transfer on all universal functions can be linearized in a rather good approximation, such that they may be represented in the form

$$F_\nu(H,M_\delta,\theta) \approx F_\nu(H,M_\delta 0)+\theta\cdot G(H,\, M_\delta). \tag{75}$$

This result will be checked after the completion of the final evaluation.

4.4. Examples—Flat Plate

For the sake of completeness we present here again an important result obtained for the laminar case with insulated wall, figures 5 and 6. Pohlhausen's method based only upon the momentum integral condition (49) and the wall condition (13) gives $H=1.572=$const., for the case of $dp/dx=0$ at each Mach number. The consideration of the energy integral

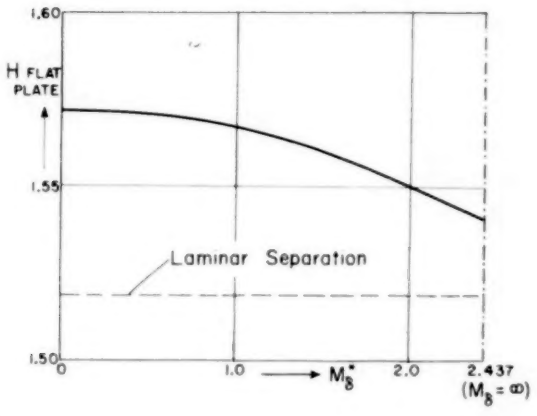


FIGURE 5. *Shape parameter H of the laminar velocity profile at a flat plate as a function of the Mach number* M_δ^*.

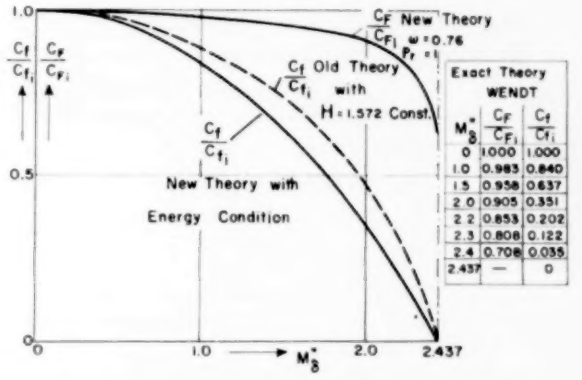


FIGURE 6. *Friction coefficient ratios* c_f/c_{f_i} *(local at the same Reynolds number* R_{δ_2}*) and* c_f/c_{f_i} *(averaged) as a function of the Mach number* M_δ^*.

Laminar boundary layer.

condition (53) instead of eq (13) gives a variation of H with M_δ^* and a corresponding change in the behavior of the friction coefficients c_f/c_{fi} (local at same values R_{δ_2}) and C_F/C_{Fi} (averaged) with the Mach number M_δ^*. It is very interesting that the new behavior of c_f/c_{fi} (M_δ^*) is almost coincident with the results of an exact theory of H. Wendt [8]. This result shows the effectiveness of the energy integral condition. The related results for the compressible turbulent boundary layer with and without heat transfer are plotted in figures 7 and 8. The agreement with available experimental data is seen to be good.

Figures 9 and 10 are concerned with the question of how the turbulent separation point in a retarded flow of the type

$$\frac{u_\delta}{u_\infty}=\frac{M_\delta^*}{M_\infty^*}=1-10x/l$$

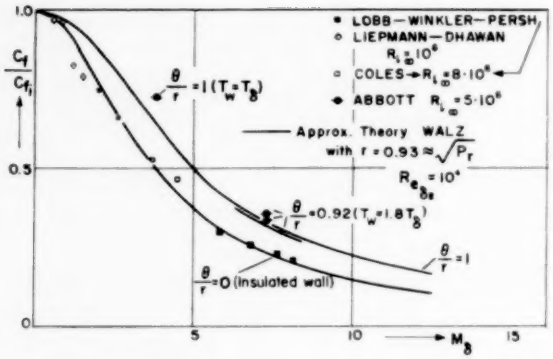

LOBB—WINKLER—PERSH.
LIEPMANN—DHAWAN
COLES → $R_{L\infty} = 8 \cdot 10^6$
ABBOTT $R_{L\infty} = 5 \cdot 10^6$
Approx. Theory WALZ
with $r = 0.93 \approx \sqrt{P_r}$
$R_{\theta_{\delta^*}} = 10^4$
$\frac{\theta}{r} = 1 \; (T_w = T_\delta)$
$\frac{\theta}{r} = 0.92 \; (T_w = 1.8 T_\delta)$
$\frac{\theta}{r} = 1$
$\frac{\theta}{r} = 0$ (insulated wall)
$\frac{c_f}{c_{f_i}}$
M_δ

FIGURE 7. *Local friction coefficient ratio c_f/c_{f_i} (at the same Reynolds number R_{δ_2}) of a flat plate as a function of the Mach number M_δ with different values of the heat transfer parameter θ/r.*

Turbulent boundary layer.

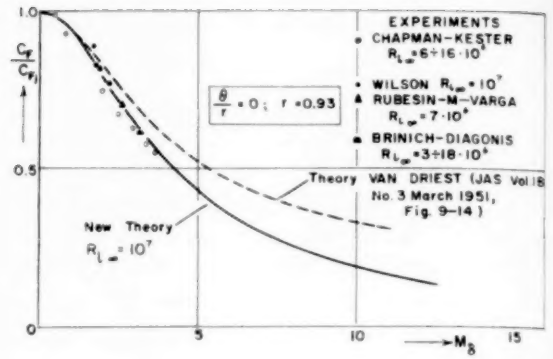


FIGURE 8. *Averaged friction coefficient ratio c_f/c_{f_i} of a flat plate as a function of the Mach number M_δ for insulated wall ($\theta/r = 0$).*

Turbulent boundary layer.

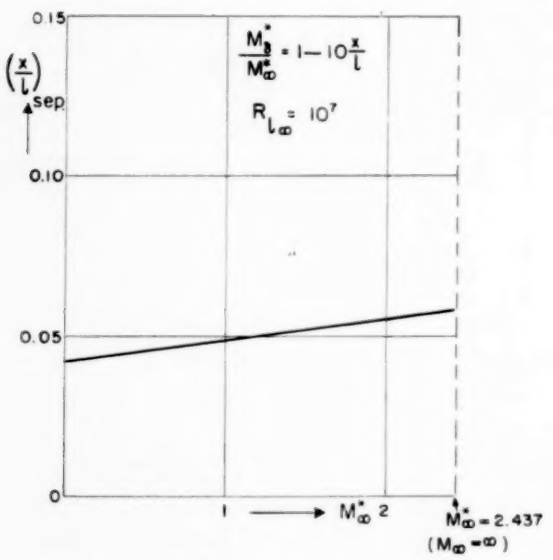
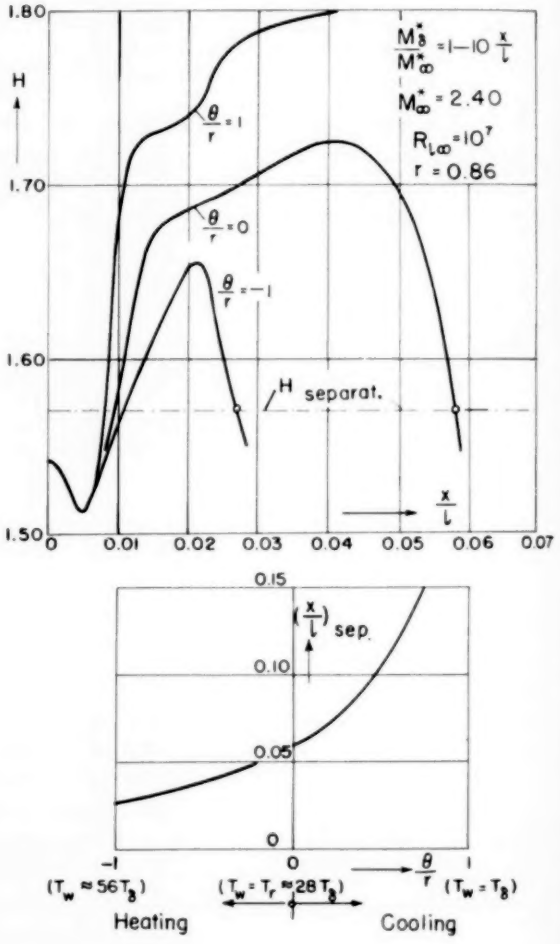


FIGURE 9. *Influence of the Mach number M^*_∞ on the separation point, insulated wall ($\theta = 0$).*

Turbulent boundary layer separation in a retarded flow of the type $\frac{M^*}{M^*_\infty} = 1 - 10\frac{x}{l}$.



FIGURES 10 and 11. *Influence of heat transfer on the separation point $\left(\frac{x}{l}\right)_{sep}$ at a constant Mach number $M^*_\infty = 2.40$ ($M_\delta \approx 12$).*

Turbulent boundary layer separation in a retarded flow of the type $\frac{M^*}{M^*_\infty} = 1 - 10\frac{x}{l}$.

is influenced by the Mach number and by heat transfer. From figure 9 it is seen that the separation point $(x/l)_{sep}$ is almost independent of variation in the Mach number between $M_\infty^*=0$ to 2.40 ($M_\infty=0$ to 12). Figure 10 shows that by cooling the wall in such a way that the wall temperature T_w is kept equal to $T_\delta\left(\frac{\theta}{r}=1\right)$, the separation is completely prevented even for the high value $M_\infty^*=2.40$ ($M_\infty=12$). On the contrary, heating of the wall moves the separation point upstream. For the theroetical results of the figures 9 and 10, no experimental confirmation is yet available.

5. Conclusions

A survey of approximation theories for compressible laminar and turbulent boundary layers with heat transfer at the wall and arbitrary pressure gradient in the flow direction shows that only a two-equation method has been used up to the present, although an infinite number of equations is available. A new derivation for the two infinite systems of integral conditions based upon the partial differential equations for the velocity and temperature field is given.

An investigation of the significance of the equations with higher order shows that their property of describing typical boundary effects vanishes more and more with increasing number ν. Hence, it is not surprising that the results obtained from the two-equation method with $\nu=0$ in both the systems (11) and (40) are good. However, the consideration of one or two equations more would be desirable, at least for a check of the results of the two-equation method in important special cases.

Preliminary investigations have to show how the higher ν-order integral terms containing the shear stress distribution τ/τ_w can be evaluated in the case of compressible turbulent boundary layers with heat transfer. It has been found that this is possible if the power of the Reynolds number is chosen to be equal for both the wall friction and dissipation law. We have chosen 0.2. In spite of this simplification in these two empirical laws, they still fit the data rather well. Further work on this matter is in progress.

This two-equation method now available with preliminary evaluated universal integral terms has been applied to the study of the dependence of the friction coefficient on compressibility and heat transfer. Good agreement with experimental data has been found.

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WASHINGTON, D.C. (Paper 63B1–8)

Publications of the National Bureau of Standards

Selected Abstracts

Mechanized computation of thermodynamic tables at the National Bureau of Standards, J. Hilsenrath and J. H. Wegstein, *Thermodynam. Properties of Fluids, London, July 10–12, 1957, Proc. Inst. of Mech. Engrs. (London) (1958).*

A high speed computing routine is described which computes the chemical composition and certain thermodynamic properties of homogeneous gaseous systems containing up to 5 elements. The procedure, which is applicable over a wide temperature range in which both dissociation and ionization effects are allowed for, permits the computation of mole fractions, compressibility factor, internal energy, enthalpy and entropy of the equilibrium mixtures at specified temperatures and at specified densities or pressure intervals. Auxiliary routines for the IBM 704 are used to compute the specific heats, specific heat ratios, isentropic expansion coefficient and the sound velocity. A brief description is given of the library of thermodynamic tables that provides the input data for these calculations. An appendix to this paper gives an indexed bibliography of existing thermodynamic functions for the products of a wide variety of combustion reactions.

Further identities and congruences for the coefficients of modular forms, M. Newman, *Can. J. Math.* **10,** *577 (1958).*

Let $p_r(n)$ be the coefficient of x^n in $\Pi(1-x^n)^r$. Let r be odd, $0<r<24$. Suppose p is a prime such that $p>3$ when $(r,3)=1$ and $p>2$ when $3|r$. Then it is shown that for all integral n,

$$p_r(np^2+r\nu)-\gamma_n p_r(n)+p^{r-2}p_r\left(\frac{n-r\nu}{p^2}\right)=0,$$

where $\gamma_n=c-\dfrac{(r\nu-n)}{p}\,p^{(r-3)/2}\,a$, and c and a are certain constants.

A fractional factorial experiment of the mixed 2^m3^n series, W. S. Connor, *Proc. 14th Ann. Clin. Rochester Soc. for Quality Control (Feb. 1958).*

The paper is concerned with a fractional factorial design for three factors each of which has two levels and two factors each of which has three levels. The complete factorial consists of 72 treatment combinations but the fractional factorial of only 36 treatment combinations. An example is worked out.

A note on confidence intervals in regression problems J. Mandel, *Ann. Math. Statist. No. 3,* **29,** *892 (Sept. 1958).*

A method is presented for the construction of jointly valid confidence intervals for arbitrary real functions of the coefficients of a multiple linear regression. In the case of a single independent variable, x, the problem of constructing jointly valid confidence intervals for the x-values corresponding to a set of "future" measurements of the dependent variable is also discussed.

Harmonic generation with ideal rectifiers, C. H. Page, *Proc. I.R.E., No. 10,* **46,** *1738 (Oct. 1958).*

It is shown that the nth harmonic cannot be generated with an efficiency exceeding $1/n^2$. Of the power converted to dc and harmonics, at least 75 percent is dc dissipation, and this cannot be reduced by an arrangement of selective circuits.

Two forms of mathematical induction, A. Schach, *Math. Mag., No. 2,* **32,** *83 (Nov.–Dec. 1958).*

It is shown that two principal forms of (finite) mathematical induction:

(I) If $P(1)$ and if, for all pos. int. n, $P(n)$ implies $P(n+1)$, then $P(n)$ for all pos. int. n; and

(II) If $P(1)$ and if, for all pos. int. n, $P(1)\&P(2)\&\ldots\&P(n)$ implies $P(n+1)$, then $P(n)$ for all pos. int. n

are equivalent; and that any demonstration based on one of them can be recast so that it is based on the other instead.

The evaluation of matrix inversion programs, M. Newman and J. Todd, *J. Soc. Indust. Appl. Math., No. 4,* **6,** *466 (Dec. 1958).*

A set of thirteen test matrices typical of those occurring in practice were inverted on SEAC, the 704, and various other machines. The results were compared with the theoretical inverses and various error estimates were computed.

Un nouveau critere d'univalence des transformations dans un R^n, M. A. Ostrowski, *Compt. rend. acad. sci.* **248,** *348 (Jan. 1959).*

A criterion of univalence of an n-dimensional transformation, given in a former communication, is generalized; the possibility of further generalizations and refinements is discussed.

On the convergence of the Rayleigh quotient iteration for the computation of the characteristic roots and vectors. II, A. M. Ostrowski, *Arch. Rational Mech. Anal. No. 5,* **2,** *423 (1959).*

Further asymptotic properties of the Rayleigh quotient algorithm are derived, and numerical results for some special matrices are given.

A note of the computation of χ^2, M. G. Natrella, *Am. Statistician, No. 1,* **13,** 20 *(Feb. 1959).*

There is a useful, but not widely-known short cut available for computing χ^2 for the comparison of two observed frequency distributions with equal total counts. The advantages are that it does not require computing the expected frequencies, avoids problems of rounding the deviations, and requires only one division per class instead of one division per cell. It can be extended to three observed distributions, but loses its computational advantages for more than three.

Mode expansion in the low-frequency range for propagation through a curved stratified atmosphere, H. Bremmer, *J. Research NBS* **63D,** *No. 1, 75 (1959).*

This expansion is particularly useful when considering ionospheric propagation at low frequencies. The complex problem dealing with two media, viz. a homogeneous earth and a surrounding stratified atmosphere, leads to intractable expressions. However, as the influence of the earth may be accounted for by an approximate boundary condition at the earth's surface, the problem is then reduced to that of the outer medium only. The coefficients of the mode expansion for this simplified problem will be derived while taking into account the earth's curvature; however, the latter proves to be negligible under very general conditions. The expansion to be derived is wanted in particular when studying the influence of a gradual transition in the electron density with height at the lower edge of the ionosphere.

On the theory of fading properties of a fluctuating signal imposed on a constant signal, H. Bremmer, *NBS Circ.* **599** *(May 1959).*

This Circular is a theoretical mathematical consideration of the varying strength of a radio signal consisting of two components: a constant signal, and a fluctuating signal of the same frequency which has reached the receiver by a different path. The properties investigated for the composed signal are the distribution functions of both the amplitude and phase, as well as the average number of crossings for either quantity through any given level.

The first part of this Circular deals with the idealized case, where the fluctuating signal is considered to be "random." The second part is a more practical consideration of the characteristics of the composite signal when the fluctuating signal is a "quasi-random" result of first order scattering in a turbulent atmosphere. In the two extreme cases, where the received signal is composed entirely of either the constant, or line-of-sight, signal, or the fluctuating, reflected signal, the complicated general formulas are simplified.

This Circular will be of interest to those who deal with the various types of equipment and effects which pertain to the fading properties of this composite signal.

The mean deviation of the Poisson distribution, E. L. Crow, *Biometrika, pts. 3 and 4* **45,** *556–559 (Dec. 1958).*

The mean deviation of the Poisson distribution is easily determined to be $2mY$, where m is the mean of the distribution and Y is the maximum value of the individual Poisson probabilities. The ratio, $R(m)$, of the mean deviation to the standard deviation is shown to be a damped oscillation as a function of m, converging rapidly toward the limiting normal value $\sqrt{2/\pi}$. Furthermore $R(m)$ takes on the normal value twice between every pair of successive non-negative integers. Thus $R(m)$ is a poor criterion for distinguishing the Poisson and normal distributions.

A property of additively closed families of distributions, E. L. Crow, *Ann. Math. Stat. No. 3,* **29,** *892–897 (Sept. 1958).*

Consider a one-parameter additively closed family of univariate cumulative distribution functions $F(x; \lambda)$ (H. Teicher, *Ann. Math. Stat.*, Vol. 25 (1954), pp. 775–778). Let three cumulants with orders in arithmetic progression exist and be nonzero. If all three orders are even, or if the first order is odd, it is also required that $F(x; \lambda)=0$ for $x<0$ and $F(x; \lambda) > 0$ for $x > 0$. Consider linear combinations, with real, nonzero coefficients, of a finite number of independent variables with distributions in the family. It is proved that the only such linear combinations whose distributions are also in the family are those with coefficients unity. The additively closed families having this property may be called *strictly additively closed.* It can be shown that (one-parameter) additively closed stable families of distributions (normal and Cauchy in particular) with characteristic functions continuous in λ are not strictly additively closed, while Poisson, generalized Poisson, binomial, and gamma families are strictly additively closed.

Other NBS Publications

Journal of Research Section A. Physics and Chemistry, Volume 63A, No. 1, July–August 1959. *70 cents. Approximate issue date, August 1, 1959.*

Description and analysis of the first spectrum of iodine. C. C. Kiess and C. H. Corliss.

CH in the solar spectrum. Charlotte E. Moore and Herbert P. Broida.

Infrared studies in the 1- to 15-micron region to 30,000 atmospheres. C. E. Weir, E. R. Lippincott, A. Van Valkenburg, and E. N. Bunting.

Phosphinoborine compounds: mass spectra and pyrolysis. Leo A. Wall, Sidney Strauss, Roland E. Florin, Fred L. Mohler, and Paul Bradt.

Experimental and theoretical study of kinetics of bulk crystallization in poly(chlorotrifluoroethylene). John D. Hoffman, James J. Weeks, and W. M. Murphey.

Journal of Research Section C. Engineering and Instrumentation, Volume 63C, No. 1, July–September 1959. *75 cents. Approximate issue date, August 15, 1959.*

Creep of cold-drawn nickel. William D. Jenkins and Carl R. Johnson.

Friction and endurance of prelubricated and unlubricated ball bearings at high speeds and extreme temperatures. Hobart S. White.

Effect of light and water on the degradation of asphalt. L. R. Kleinschmidt and H. R. Snoke.

Current and potential relations for the cathodic protection of steel in a high resistivity environment. W. J. Schwerdtfeger.

A tilting air-lubricated piston gage for pressures below one-half inch of mercury. U. O. Hutton.

Compact multianvil wedge type high-pressure apparatus. E. C. Lloyd, U. O. Hutton, and D. P. Johnson.

A coulometric-titration coulometer. Stanley W. Smith and John K. Taylor.

Electron beam magnetometer. L. Marton, Lewis B. Leder, J. W. Coleman, and D. C. Schubert.

A refined X-band microwave microcalorimeter. Glenn F. Engen.

Journal of Research Section D. Radio Propagation, Volume 63D, No. 1, July–August 1959. *70 cents. Approximate issue date, July 15, 1959.*

Preliminary results of the National Bureau of Standards radio and ionospheric observations during the International Geophysical Year. David M. Gates.

Origin of [OI] 5577 in the airglow and the aurora. Franklin E. Roach, James W. McCaulley, and Edward Marovich.

Comparison of absolute intensities of [OI] 5577 in the auroral and subauroral zones. F. E. Roach, J. W. McCaulley, and C. M. Purdy.

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Low-frequency propagation paths in arctic areas. A. D. Watt, E. L. Maxwell, and E. H. Whelan.

A property of additively closed families of distributions, E. L. Crow, *Ann. Math. Statist. No. 3,* **29,** *892 (Sept. 1958).*

The mean deviation of the Poisson distribution, E. L. Crow, *Biometrika* **45,** *556 (1958).*

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